Vol. 79, No. 2 DUKE MATHEMATICAL JOURNAL © August 1995

RIGID PATHS OF GENERIC 2-DISTRIBUTIONS ON 3-MANIFOLDS

I. ZELENKO AND M. ZHITOMIRSKII

Introduction. Let M be a smooth connected manifold, and E a bracket-generating (=nonholonomic) k-dimensional distribution on M (a smooth k-dimensional subbundle of TM). A smooth path γ: $[\alpha, \beta] \to M$ is called admissible (or E-path or horizontal) if it is tangent to E: $\dot{\gamma}(t) \in E(\gamma(t))$ for all $t \in [\alpha, \beta]$. Given two points $a, b \in M$, denote by $\Omega_E(a, b)$ the space of all E-paths γ: $[0, 1] \to M$ joining a to b: $\gamma(0) = a$, $\gamma(1) = b$. The space $\Omega_E(a, b)$ is not empty (by the Chow theorem) and, being endowed with a natural C^1-topology, might have singular points (called abnormal paths; several equivalent definitions can be found in [1], [5], [6], [10]) and even isolated points (called rigid paths [5]). More precisely, a path $\gamma \in \Omega_E(a, b)$ is called rigid if any C^1–close enough path of $\Omega_E(a, b)$ is a smooth reparametrization of γ. An arbitrary E-path γ defined on an interval $[\alpha, \beta]$ is called rigid if some (and then any) of its smooth reparametrization belonging to $\Omega_E(\gamma(\alpha), \gamma(\beta))$ is rigid.

Admissible, abnormal, and rigid paths can be also defined in terms of control theory. Assume for simplicity that $M = \mathbb{R}^n$. Then the distribution E is generated by k smooth independent vector fields $v_1(x), \ldots, v_k(x)$, $x \in \mathbb{R}^n$, and can be interpreted as a control system

$$\dot{x} = u_1(t)v_1(x) + \cdots + u_k(t)v_k(x), \tag{0.1}$$

where $u(t) = (u_1(t), \ldots, u_k(t))$ is an arbitrary smooth vector function (called control). The E-paths are exactly the solutions of (0.1) defined on compact intervals. The solution $x_u(t)$ corresponding to a control $u(t)$ and defined on $I = [\alpha, \beta]$ is rigid if there exists $\varepsilon > 0$ such that it is a smooth reparametrization of the solution $x_w(t)$ corresponding to a control $w(t)$, provided that the solutions join the same points ($x_u(\alpha) = x_w(\alpha)$ and $x_u(\beta) = x_w(\beta)$) and $\max_{t \in I} \|u(t) - w(t)\| < \varepsilon$.

The abnormal and rigid paths attract the attention of mathematicians working in sub-Riemannian geometry, control theory, calculus of variations, and singularity theory. The study of abnormal and rigid paths became very intensive after the discovery by R. Montgomery ([8], [10], [11]) that a rigid path might be a sub-Riemannian minimizer (the shortest admissible path joining its endpoints with respect to a given sub-Riemannian metric, i.e., a metric on E; sub-Riemannian metric allows us to measure the length of any admissible path) while it does not

Received 14 November 1994.
Zhitomirskii's work supported in part by Technion VPR Fund.

satisfy the sub-Riemannian geodesic equations (Euler-Lagrange equations for the Lagrange problem in the calculus of variations). Moreover, a rigid curve might be a sub-Riemannian geodesic with respect to any sub-Riemannian metric ([6], [8], [10]).

Now that these phenomena have been discovered, the theory of abnormal and rigid paths is being intensively developed, and the main directions are as follows:

(1) The relations between the rigidity, abnormality, and one of the optimality properties. One can distinguish optimality properties which are (a) local-in-time and global-in-paths-space (like the property to be a geodesic), (b) global-in-time and local-in-paths-space [2];

(2) The description of abnormal and rigid paths of generic k-distributions on n-manifolds or of distributions of a certain special class (say, with a fixed growth vector).

Results in the first direction can be found in works [1], [2], [6], [8], [10], [11].

The present paper is a contribution to the second direction presented, at the time being, by the following results:

(1) A complete description of abnormal and locally rigid paths of Martinet distributions [6], [9], [10]. A Martinet distribution is a 2-distribution on a 3-manifold M^3 with the growth vector (2, 3) at a generic point of M^3 (points outside a smooth surface S) and the growth vector (2, 2, 3) at any point of S. The set of all Martinet distributions is open in the space of all 2-distributions on M^3, but it is not dense.

(2) A complete description of abnormal and rigid paths of Engel distributions ([5], [6]; see also [11]). An Engel distribution is a 2-distribution on a 4-manifold M^4 with the maximum growth vector (2, 3, 4) at any point. The set of Engel distributions is open in the space of all 2-distributions on M^4, but it is not dense.

(3) A complete description of abnormal and locally rigid paths of 2-distributions on a 5-manifold M^5 with the growth vector (2, 3, 4, 5) at any point ([5], [6]; see also [11]). The set of these distributions has codimension ∞ in the space of all 2-distributions on M^5.

(4) Local existence theorems for rigid paths of generic 2-distributions on a manifold M^n of dimension $n \geqslant 5$: if $p \in M^n$ is a generic point, then for any admissible direction l at p there exists a rigid path passing through p in the direction l ([5], [6]; see also [11]). Moreover, there exists a family of rigid fields of directions, i.e., a family of line subdistributions d of a 2-distribution E such that any d-path is E-rigid [12].

(5) The absence of abnormal paths of strong bracket-generating distributions (in particular, contact structures), and the absence of rigid paths passing through a generic point of a generic k-distribution on an n-manifold if $n \leqslant k + k(k-1)/2$ (see [1], [6], [8]).

The contribution of this paper is a complete and explicit description of all rigid paths of generic 2-distributions on a 3-manifold M^3. (It means that we describe all rigid paths of 2-distributions on M^3 of a certain open and dense set in the space of all 2-distributions.) As far as we know, it is the first result of this type.

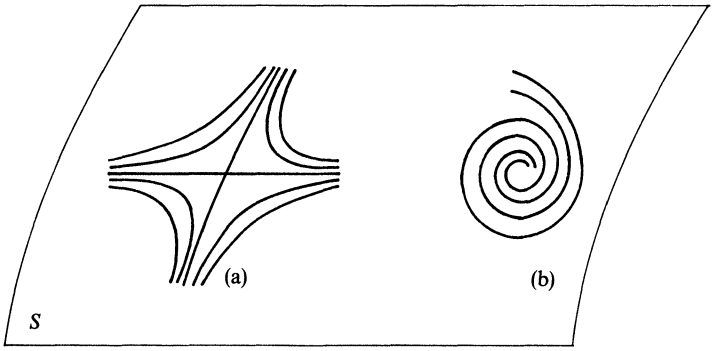

FIGURE 1. Local behaviour of the leaves of the foliation F near (a) a saddle point, (b) a focus point

The paper is organized as follows. The first section contains main results of the works [7], [13], [14] concerning local geometry and normal forms of generic 2-distributions on M^3. (We need to distinguish different types of singularities to formulate main results; the normal forms are used in our proofs.) In particular, we define the Martinet surface S—the surface of all singular points (at which a generic 2-distribution E on M^3 does not define a contact structure), and three types of points of S: transversality points (at which E is transversal to S), saddle points, and focus points. Saddle and focus points are isolated points of S at which E is tangent to S, they are singular points of the foliation $F = TS \cap E$ of S. The topological types of F near a saddle and a focus point are different (see Figure 1).

It is known that all abnormal paths live in the Martinet surface S [6], [10]. Therefore, to describe rigid paths, we have to analyze the structure of all admissible paths in the Martinet surface. We do this in Section 2. A path containing only transversality points of S is contained in a leaf of the foliation F, and therefore we have to understand the local structure of paths in S containing saddle and focus points. The foliation F is generated by a smooth vector field C on S vanishing at saddle and focus points. (We define C in the first section and call it the characteristic vector field.) For a path in the Martinet containing a saddle point p to be admissible, it must lie in one of the C-invariant manifolds (stable and unstable 1-dimensional manifolds) that meet transversally at p (see Figure 1 (a)). A harder question is about the possibility for a smooth path to contain a focus point. This question is equivalent to asking whether the length of a spiral at Figure 1 (b) is finite or infinite (with respect to some and then any metrics). This question is very important for the sub-Riemannian geometry where only arc-length-parametrizable paths are studied. To answer, we use a well-known normal form of a vector field in a plane near a focus point at which the eigenvalues are pure imaginary. (The characteristic vector field C has this property at any focus point.) We show that any solution of a system of ordinary differential equations corresponding to C, defined on $[0, \infty)$, and tending to a focus point as $t \to \infty$ has the infinite length. It follows that (a) there are no abnormal arc-length-parametrized absolutely continuous paths containing focus points, and (b) there are no rigid (and even abnormal) smooth paths containing focus points. All smooth admissible paths in S can be divided into four types: (1) paths containing no tangency points, (2) paths $\gamma: [\alpha, \beta] \to S$ such that the restriction of γ to (α, β) contains no tangency point, one of the end points $\gamma(\alpha)$, $\gamma(\beta)$ is transversality and the other is a saddle point, (3) paths $\gamma: [\alpha, \beta] \to S$ such that the restriction of γ to (α, β) contains no tangency point and both end points are saddle points, and (4) paths $\gamma: [\alpha, \beta] \to S$ for which there exists $t \in (\alpha, \beta)$ such that $\gamma(t)$ is a saddle point.

The possibility of the existence of immersed E-paths in $S(E)$ of the types (2) and (4) and nonclosed simple paths without tangency points is trivial. In Section 2 we also give examples of paths of the type (3) and of admissible *closed* paths in S (the endpoints coincide) containing transversality points only. The latter paths in our examples are immersed, but noncontractible. We also give an example of an admissible closed simple path in S that is contractible but contains two saddle points. We formulate a conjecture on the nonexistence of closed contractible (in S) abnormal paths without tangency points.

The main result is formulated in Section 3: a smooth immersed path $\gamma: [\alpha, \beta] \to M^3$ is rigid if and only if it is a path in the Martinet surface and, for any $t \in (\alpha, \beta)$, the point $\gamma(t)$ is a transversality point. It means that an immersed path is rigid if and only if it belongs to one of the types (1)–(3). Note that immersed paths of the first type might be closed or self-intersecting (if their image is contained in a cycle of the characteristic vector field), and therefore we prove the rigidity of the restriction to *any* compact interval of *any* periodic solution of the characteristic vector field.

The fact that a path in S of the type (4) is not rigid is proved in Section 4: we use a local normal form of a distribution near a saddle point to build a smooth 1-parameter deformation containing paths joining the same endpoints.

In Section 5 we introduce the notions of separating and strongly separating surface of an admissible path. We believe that these notions have independent significance. The existence of a separating surface is a stronger property than the rigidity. We show (by using normal forms) that, locally, a separating surface exists for any admissible path in S of one of the types (1)–(3). It proves the local version of the main theorem.

To prove the global rigidity of paths of the types (1)–(3), we give an invariant way of constructing strongly separating surfaces of any immersed nonclosed admissible path in S without tangency points (Section 6), and a way of pasting of separating surfaces (Section 7). We also prove that any immersed admissible path in $S(E)$ containing no tangency points has a strongly separating surface, even if it is closed or self-intersecting.

Acknowledgements. We would like to thank Andrei Agrachev and Richard Montgomery for a number of stimulating questions and discussions.

1. Geometry of singularities and normal forms. In this section we present main results of the works [7], [13], [14] on local geometry and normal forms of 2-distributions on 3-manifolds in the form convenient for the problems of the sub-Riemannian geometry.

1.1. Singular points. Let E be a smooth 2-distribution on a smooth[1] manifold M^3 (a smooth 2-dimensional subbundle of TM^3). Take a point $p \in M^3$ and its contractible neighbourhood U. Take a basis (v_1, v_2) of sections of E restricted to U. The point p is called *singular* if

$$\dim \operatorname{span}(v_1(p), v_2(p), [v_1, v_2](p)) = 2.$$

This condition can be replaced by an equivalent one, $(\omega \wedge d\omega)(p) = 0$, where ω is a nonvanishing differential 1-form on U annihilating E (i.e., annihilating any section of E).

Notation. We denote by S or $S(E)$ the set of all singular points. Sometimes we will call $S(E)$ the Martinet surface.

1.2. Regular singular points. We will say that a singular point p is *regular* if

$$j_p^1(\omega \wedge d\omega) \neq 0.$$

Here j_p^1 is the 1-jet at the point p.

PROPOSITION 1.1. *Let E be a generic 2-distribution on M^3. Then either $S(E) = \varnothing$ or any point of $S(E)$ is regular, and consequently $S(E)$ is a smooth 2-dimensional submanifold of M^3.*

Remark 1.1. As usual, when saying that some statement holds for a generic 2-distribution, we mean that, in the set of all 2-distributions, there exists an open dense subset (with respect to the Whitney topology, see [4], [7]) such that the statement is true for every 2-distribution of this subset.

1.3. Transversality and tangency singular points. Let p be a regular singular point of a 2-distribution E.

Definition. We will say that p is a transversality singular point if E is transversal to $S(E)$ at p, i.e., $E(p) + T_pS(E) = T_pM^3$. Regular singular points violating this condition will be called tangency points. In other words, a regular singular point p is a tangency point if $E(p) = T_pS(E)$.

1.4. Characteristic vector field on the Martinet surface. Assume that, in some domain $D \subset M^3$, all singular points of the distribution E are regular, and the

[1] Throughout the paper, manifolds, functions, vector fields, differential forms, distributions, etc., are assumed to be of the class C^∞, and curves and paths are assumed to be of the class C^1, unless there is an explicit mention to the contrary.

restriction of E to D can be given as a field of kernels of a nonvanishing 1-form ω. Denote by ω_1 the pullback of ω to the set $S(E) \cap D$. (This set is a smooth surface in D by Proposition 1.1.) One can prove that the Martinet surface is always orientable. Take a nondegenerated volume form μ on $S(E) \cap D$, and define a vector field C on $S(E) \cap D$ by the relation $C \rfloor \mu = \omega_1$ (i.e., $\mu(C, Y) = \omega_1(Y)$ for any vector field Y on $S(E) \cap D$).

Definition. Any vector field C defined as above will be called a characteristic vector field on the Martinet surface.

Remark 1.2. Note that C depends on the choice of ω and μ, but any change of ω or μ leads just to the multiplication of C by a nonvanishing function. This means that the module of vector fields on $S(E) \cap D$ generated by any characteristic vector field C on $S(E) \cap D$ is invariantly related to E.

PROPOSITION 1.2. *Let p be a regular singular point, let C be an arbitrary characteristic vector field on a neighbourhood $U \subset S$ of p in S, and let (v_1, v_2) be an arbitrary basis of sections of E restricted to a neighbourhood of p in M^3. The following statements are equivalent:*

(a) *p is a tangency point,*

(b) $C(p) = 0$,

(c) $\dim \operatorname{span}(v_1(p), v_2(p), [v_1, v_2](p), [v_1, [v_1, v_2]](p), [v_2, [v_1, v_2]](p)) = 2$.

Of course, condition (b) is independent of the choice of C, and condition (c) is independent of the choice of the basis of sections (v_1, v_2).

PROPOSITION 1.3. *Let p be a tangency point, and let C be an arbitrary characteristic vector field on a neighbourhood of p in the Martinet surface. The sum of the eigenvalues of the linearization of C at p is equal to 0.*

Note that if C and $\tilde{C}$ are two characteristic vector fields, and if $\lambda_{1,2}$ and $\tilde{\lambda}_{1,2}$ are their eigenvalues at the same tangency point, then there exists a real factor $k \neq 0$ such that $k\lambda_1 = \tilde{\lambda}_1, k\lambda_2 = \tilde{\lambda}_2$ or $k\lambda_1 = \tilde{\lambda}_2, k\lambda_2 = \tilde{\lambda}_1$.

1.5. Hyperbolic, elliptic, and parabolic tangency points. Proposition 1.3 allows us to distinguish three types of tangency points. Let p be a tangency point, and let C be a characteristic vector field on a neighbourhood of p in the Martinet surface. We will say that p is hyperbolic if the eigenvalues of C at p are real and nonzero (i.e., ± 1 up to a real nonzero factor), it is elliptic if they are pure imaginary (i.e., $\pm\sqrt{-1}$ up to a real nonzero factor), and it is parabolic if both eigenvalues are equal to 0.

By Proposition 1.3, any tangency point is either hyperbolic, or elliptic, or parabolic.

PROPOSITION 1.4. *Let E be a generic 2-distribution. Then either (1) there are no tangency points or (2) any tangency point is either hyperbolic or elliptic (there are no parabolic tangency points), and then the tangency points are isolated.*

1.6. Foliation on the Martinet surface. Denote by $T(E)$ the set of all tangency points of a distribution E. Assume that all singular points of E are regular. At each point $p \in S(E) - T(E)$, the direction $l_p = E(p) \cap T_pS(E)$ is invariantly related to E. We obtain a field of directions $F = \{l_p\}$ on $S(E) - T(E)$ (or, equivalently, a line subbundle of $T(S(E) - T(E))$. It induces a foliation of $S(E) - T(E)$ by curves which we also denote by F. The tangency points are singular points of F. It is clear that if C is a characteristic vector field on a domain $U \subset S(E)$, then F restricted to U is generated by C in the sense that $C(p) \in l_p$ for any transversal point $p \in U$. Therefore, the behaviour of the leaves of F near the tangency points is defined by the local phase portrait of C. (The leaves of F are the phase curves of C.)

The phase portrait of any characteristic vector field C near a hyperbolic tangency point is a saddle (see Figure 1 (a)), since the eigenvalues of C at p are real and of different signs. The phase portrait of C near an elliptic tangency point depends on the nonlinear part of the Taylor series of C at p. It turns out that it is always a weak nondegenerated focus (Figure 1 (b)).

1.7. Weak nondegenerated focus. The definition of a weak nondegenerated focus is as follows. Let C be a vector field on a 2-manifold S having pure imaginary eigenvalues $\pm\alpha\sqrt{-1}$ at a point $p \in S$. There exist local coordinates x, y on S (near p) such that the 3-jet of C at p is as follows:

$$j_p^3C = (\kappa x_1 R^2 + (\alpha + bR^2)x_2)\frac{\partial}{\partial x_1} + (-x_1(\alpha + bR^2) + \kappa x_2 R^2)\frac{\partial}{\partial x_2},$$

where $R^2 = x_1^2 + x_2^2$. This means that in the polar coordinates R, φ ($x = R\cos\varphi$, $y = R\sin\varphi$), the restriction of the field C to a small enough neighbourhood of p is given by the system of ordinary differential equations of the form

$$\dot{R} = \kappa R^3 + o(R^3), \qquad \dot{\varphi} = -\alpha + o(R). \tag{1.1}$$

Definition. We say that the phase portrait of C near p is a weak nondegenerated focus if the eigenvalues of C at p are pure imaginary, and the parameter κ in the normal form (1.1) is not 0.

If the phase portrait is a weak nondegenerated focus, then the phase curves of C located near p are spirals (Figure 1 (b)). If $\kappa < 0$, then p is an asymptotically stable singular point of C, if $\kappa > 0$ for the field $-C$ (see [3]).

PROPOSITION 1.5. *Let p be an elliptic tangency point of a 2-distribution E. Then the local (near p) phase portrait of any characteristic vector field C is a weak nondegenerated focus.*

1.8. Terminology: saddle and focus points. Proposition 1.5 allows us to call elliptic tangency points *focus points*. We will use this term, as well as the term *saddle point* for a hyperbolic tangency point.

1.9. The growth vector. Given a 2-distribution E on a neighbourhood of a point p generated by vector fields v_1 and v_2, define the sequence $V_1, V_2, \ldots$ of modules of vector fields: V_1 is generated by v_1 and v_2, and V_{j+1} is generated by the vector fields of V_j and all the Lie brackets $[v, \tilde{v}]$, where $v \in V_1$, $\tilde{v} \in V_j$. Denote by $a_j(p)$ the dimension of the subspace $\{v(p), v \in V_j\} \subset T_pM^3$. If E is a bracket-generating distribution, then, by definition, for any $p \in M^3$ there exists an s such that $a_1(p) = \cdots = a_s(p) = 2$, $a_{s+1}(p) = a_{s+2}(p) = \cdots = 3$. The tuple $(2, 2, \ldots, 2, 3)$ (s times the number 2) is called the growth vector of E at the point p.

It follows from the definition of a singular point and from Proposition 1.2 that

(a) the growth vector at any nonsingular point is (2, 3),

(b) the growth vector at any transversality singular point is (2, 2, 3).

PROPOSITION 1.6. *The growth vector at any saddle or focus point is* (2, 2, 2, 3).

Remark 1.3. The growth vector at a generic parabolic tangency point (i.e., parabolic tangency points of generic 1-parameter families of 2-distributions) is also (2, 2, 2, 3); this means that the three types of the tangency points cannot be distinguished in terms of the growth vector.

1.10. Normal forms. It follows from the classical Darboux theorem that, near a nonsingular point of a 2-distribution E, there exist coordinates x, y, z in which E is generated by vector fields $v_1 = \partial/\partial x$, $v_2 = \partial/\partial y + x(\partial/\partial z)$.

PROPOSITION 1.7. *Let p be a transversality singular point of E. Then there exist coordinates x, y, z near p such that the germ of E at p is generated by vector fields*

$$v_1 = \frac{\partial}{\partial x}, \qquad v_2 = \frac{\partial}{\partial y} + x^2\frac{\partial}{\partial z}. \tag{1.2}$$

Note that if E is generated by vector fields (1.2), then the Martinet surface is the (y, z)-plane and any characteristic vector field has the form $\partial/\partial y$, up to multiplication by a nonvanishing function.

PROPOSITION 1.8. *Let p be a saddle point of E. Then there exist coordinates x, y, z near p such that the germ of E at p is generated by vector fields*

$$v_1 = \frac{\partial}{\partial x}, \qquad v_2 = \frac{\partial}{\partial z} + (xy + x^2 z + bx^3 z^2)\frac{\partial}{\partial y}, \tag{1.3}$$

where b is a parameter (distinguishing nonequivalent germs).

If E is generated by vector fields (1.3), then the Martinet surface is given by the equation $y + 2xz + 3bx^2z^2 = 0$. The functions x and z are coordinates on the Martinet surface, and any characteristic vector field has the following form in these coordinates (up to multiplication by a nonvanishing function):

$$C = (2x + (6b - 1)x^2 z - 2bx^3 z^2)\frac{\partial}{\partial x} - (2z + 6bxz^2)\frac{\partial}{\partial z}. \tag{1.4}$$

The invariant submanifolds $L_{1,2}$ of any characteristic vector field on $S(E)$ containing the saddle point $p = (0, 0, 0)$ are given by the equations $x = y = 0$ and $z = y = 0$ (the z-axis and the x-axis). For the characteristic vector field (1.4), the z-axis is the stable manifold, and the x-axis is the unstable manifold.

Of course, there is no qualitative difference between the invariant manifolds L_1 and L_2 (if we do not fix a characteristic vector field), and the following slight generalization of Proposition 1.8 holds.

PROPOSITION 1.8′. *Let p be a saddle point of E. Let L be one of the invariant manifolds $L_{1,2}$ of a characteristic vector field containing p. Then (a) there exist coordinates x, y, z near p such that E is generated by vector fields (1.3) and L is the x-axis, (b) there exist coordinates x, y, z near p such that E is generated by vector fields (1.3) and L is the z-axis.*

Proof. In coordinates of the normal form (1.3), E is given by a Pfaffian equation

$$\omega = dy - (xy + x^2z + bx^3z^2)\,dz = 0. \tag{1.5}$$

It suffices to show that there exists a change of coordinates which replaces the x- and the z-axis and preserves (1.5), i.e., preserves ω up to the multiplication by a nonvanishing function. Such a change can be constructed as follows. Change x by z and z by x, and then change y by $y + xyz$ and divide the obtained 1-form by $(1 + xz)$. We obtain a 1-form with the 3-jet $dy + xy\,dz + xz^2\,dx$ which can be reduced to the 3-jet of ω by the change of $y \to -y + x^2z^2/2$, $z \to -z$ and multiplication by (-1). Therefore there exists a change replacing the x- and z-axis and bringing (1.5) to the form $\omega_1 = 0$, where $j^3\omega_1 = j^3\omega$. Now we can use a particular result of [14]: any Pfaffian equation given by a 1-form with the 3-jet $j^3\omega$ is reducible to (1.5) by a smooth change of coordinates with the identity of linear approximation. □

The following theorem gives an exact normal form near a focus point.

PROPOSITION 1.9. *Let p be a focus point of E. Then there exist coordinates x, y, z near p such that the germ of E at p is generated by vector fields*

$$v_1 = \frac{\partial}{\partial x}, \qquad v_2 = \frac{\partial}{\partial z} + (xy + x^3/3 + xz^2 + bx^3z^2)\frac{\partial}{\partial y},$$

where b is a parameter distinguishing nonequivalent germs.

2. Admissible curves in the Martinet surface. Beginning from this section, we assume that a 2-distribution E on M^3 satisfies the following two genericity conditions:

$$\text{Any singular point of } E \text{ is regular (see Section 1.1),} \tag{G1}$$

$$\text{There are no parabolic tangency points (see Section 1.5).} \tag{G2}$$

We keep the notations of Section 1 and denote by $S = S(E)$ the set of all singular points (the Martinet surface) and by $T = T(E)$ the set of all tangency points (consisting, under the genericity conditions (G1) and (G2), of saddle and focus points).

It is known that all abnormal (and therefore all rigid) smooth (and even absolutely continuous) paths lie in the Martinet surface (see [6], [10]). Therefore, to describe all rigid paths, we have to analyze the structure of all admissible paths in S.

2.1. Admissible paths in $S(E) - T(E)$. The following proposition is an easy corollary of the definition of a characteristic vector field and the foliation F on the Martinet surface.

PROPOSITION 2.1. (1) *A path in $S(E) - T(E)$ is admissible if and only if its image is contained in a single leaf of the foliation F.*

(2) *Let C be an arbitrary characteristic vector field on a domain $U \subset S(E)$, free of tangency point. Let γ be an arbitrary immersed admissible path*[2] *in U. There exists a smooth reparametrization $\tilde{\gamma}$ of γ such that $\tilde{\gamma}$ is a solution of the equation*

$$\dot{\gamma}(t) = C(\gamma(t)). \tag{2.1}$$

Proof. The first statement follows from the definitions. To prove the second one, let us note that an immersed admissible path γ in U satisfies the differential equation $\dot{\gamma}(t) = c(t)C(\gamma(t))$, where C is an arbitrary characteristic vector field, and $c(t)$ is a nonvanishing function (depending on C). Assume that γ is defined on an interval I. Let $\Phi(t) = \int_0^t c(s)\,ds$, $\tilde{I} = \Phi(I)$. Since Φ is either an increasing or decreasing function, the curve $\tilde{\gamma}$ defined on $\tilde{I}$ by the relation $\tilde{\gamma}(\Phi(t)) = \gamma(t)$, $t \in I$ is a reparametrization of γ, and it satisfies the equation (2.1). □

2.2. Admissible paths in $S(E)$ containing a saddle point. Let $p \in S(E)$ be a saddle point, and let C be an arbitrary characteristic vector field on a neighbourhood U of p in $S(E)$. Denote by $L_{1,2}$ the two C-invariant 1-dimensional submanifolds of U meeting transversally at p. The next observation also follows from the definition of a characteristic vector field and a saddle point.

PROPOSITION 2.2. *A C^1 path in the neighbourhood U containing the saddle point p is admissible if and only if its image is contained in one of the submanifolds $L_{1,2}$.*

2.3. Admissible paths in $S(E)$ containing a focus point. Let $p \in S(E)$ be a focus point. Take any characteristic vector field C for which p is a weak nondegenerated stable focus. (We can take an arbitrary characteristic vector field and change its sign if necessary; see Section 1.7.) Take an attracting neighbourhood U of p in $S(E)$ (a neighbourhood such that any solution of the equation (2.1), starting at a point of U, tends to p as $t \to \infty$). By Proposition 2.1, we have the following statement.

[2] Here and below, by an immersed path we mean a C^1 mapping $\gamma\colon I = [\alpha, \beta] \to M$ such that $\dot{\gamma}(t) \neq 0$ for any t of the interval I *including* the points $t = \alpha$ and $t = \beta$.

PROPOSITION 2.3. *Any solution in U of (2.1) is an admissible curve, and any admissible curve $\gamma\colon [\alpha, \infty) \to U$ such that $\lim_{t\to\infty} \gamma(t) = p$ and $\dot\gamma(t) \neq 0$, $t \in [\alpha, \infty)$ appears in this way, up to a reparametrization.*

It turns out that the length of the spiral at Figure 1 (b) is infinite. (It is clear that if the length of a curve is infinite with respect to some metric, then the same is true for any metric.) More precisely, the following statement holds.

LEMMA 2.1. *Let C be a characteristic vector field for which a focus point p is stable. Let U be an attracting neighbourhood of p in $S(E)$. The length of any solution $\gamma\colon [0, \infty) \to U$ of the equation (2.1) is infinite.*

The following statements, which we formulate as theorems, are corollaries of this lemma and Proposition 2.3.

THEOREM 2.1. *Any admissible absolutely continuous curve*[3] *in $S(E)$ of finite length contains no focus points.*

THEOREM 2.2. *Any C^1 admissible path in $S(E)$ contains no focus points.*

THEOREM 2.3. *Any absolutely continuous arc-length-parametrized abnormal path contains no focus points.*

Proof of Lemma 2.1. We can use the polar coordinates in which C is given by the system of ordinary differential equations (1.1). It suffices to prove that if $\kappa < 0$ and $\alpha \neq 0$, then any solution $\gamma\colon [0, \infty) \to \mathbb{R}^2$ of the equation (2.1) has infinite length with respect to the metrics $dR^2 + R^2\, d\varphi^2$ (corresponding to the Euclidean metrics $dx_1^2 + dx_2^2$). In other words, we have to prove the divergence of the integral

$$\int_0^\infty \sqrt{\dot R^2(t) + R^2(t)\dot\varphi^2(t)}\, dt \tag{2.2}$$

for any solution $R(t)$, $\varphi(t)$, defined on $[0, \infty)$, of the system (1.1). For t big enough, we have

$$\begin{aligned}(\dot\varphi(t))^2 &> \alpha^2/4,\\ \dot R(t) &< \frac{\kappa}{2} R^3(t) < 0.\end{aligned} \tag{2.3}$$

The latter inequality implies that, for t big enough, there exist positive constants C_1 and C_2 such that

$$R^2(t) > \frac{C_1}{1 + C_2 t}, \tag{2.4}$$

[3] A curve is defined on either a compact or open interval (may be infinite); a path is a curve defined on a compact interval.

and the divergence of the integral (2.2) follows from (2.3) and (2.4). (The function in (2.2) behaves as $1/\sqrt{t}$ as $t \to \infty$.) □

2.4. Saddle connections. We have shown that any admissible path in $S(E)$ contains no focus points, and that it may contain saddle points. We have also shown that there exist immersed paths starting, ending, or passing through a saddle point. One can ask if it is possible that an immersed admissible path in $S(E)$ contains two or more saddle points. The following example gives the positive answer: the saddle connections in the Martinet surface are possible.

Example 2.1. Consider a 2-distribution E on $\mathbb{R}^3$ generated by vector fields $\partial/\partial x$ and $\partial/\partial z + (xy + f(x)z)(\partial/\partial y)$, where $f(x)$ is a smooth function such that $f'(x)$ and $f''(x)$ do not vanish simultaneously. The Martinet surface S is given by the equation $y = f'(x)z$, and a point $p \in S$ is a tangency point if and only if it has coordinates $(x_0, 0, 0)$ such that $f'(x_0) = 0$. Any tangency point is a saddle point; therefore E satisfies the genericity conditions (G1) and (G2). A path $t \to (t, 0, 0)$ defined on $[\alpha, \beta]$ is an admissible path in S, and if $f'(\alpha) = f'(\beta) = 0$, then this path joins the saddle point $(\alpha, 0, 0)$ to the saddle point $(\beta, 0, 0)$. (One can take, for example, $f(x) = x^3/3 - x$, $\alpha = -1$, $\beta = 1$.)

Remark 2.1. We have shown the possibility of saddle connections for distributions satisfying the genericity conditions (G1) and (G2). On the other hand, one can prove that, in the space of all 2-distributions on M^3, there exists an open and dense set such that, for any 2-distribution of this set, there are no admissible curves in the Martinet surface containing more than one saddle point.

2.5. Closed paths in the Martinet surface. One also can ask about the possibility of existence of immersed abnormal *closed* paths (paths γ: $[\alpha, \beta] \to S$ such that $\gamma(\alpha) = \gamma(\beta)$).A simple example of a distribution without tangency points for which closed abnormal paths exist is as follows.

Example 2.2. Take a torus $T^2 = \{(\phi, \psi) \bmod 2\pi\}$, and let $M = T^2 \times \mathbb{R} = \{(\phi, \psi, z)\}$. Consider a globally defined 2-distribution on M generated by vector fields $\partial/\partial z$ and $\partial/\partial\phi + (z^2 + A(\phi, \psi))(\partial/\partial\psi)$, where A is a periodic function. The Martinet surface S is given by the equation $z = 0$, i.e., $S = T^2$. It contains no tangency points. The vector field $C = \partial/\partial\phi + A(\phi, \psi)(\partial/\partial\psi)$ is a characteristic vector field on S. One can choose A so that all maximal phase curves of C are cycles (for example, $A = 0$) or so that C has a finite number of limit cycles (for example, $A = \sin\psi$). Any immersed parametrization of any cycle is an immersed abnormal closed path.

In this example the Martinet surface is a torus, and the closed paths are noncontractible.[4]

The following example gives a smooth closed simple abnormal path which is contractible, but it contains two saddle points.

[4] Here and below, by noncontractible we mean noncontractible in the Martinet surface.

Example 2.3. Consider a distribution E on $\mathbb{R}^3$ given by the differential 1-form $dy - (xy + A(x, z))\, dz$, where $A = (x^2 + z^2 - 1)^2$. The Martinet surface S is given by the equation $y = -4x(x^2 + z^2 - 1)$. A characteristic vector field can be defined on $\mathbb{R}^3$: in the coordinates x, z on S, it is as follows:

$$C = \left(\frac{\partial^2 A}{\partial x \partial z} + A - x\frac{\partial A}{\partial x}\right)\frac{\partial}{\partial x} - \frac{\partial^2 A}{\partial x^2}\frac{\partial}{\partial z}. \tag{2.5}$$

The vector field C has four singular points:

$$a_1 = (0, 1), \qquad a_2 = (0, -1), \qquad a_3 \approx (0.55, -0.25), \qquad a_4 \approx (-0.55, 0.25).$$

A simple calculation shows that the eigenvalues of C at a_1 and a_2 are ± 1, and at a_3 and a_4 are $\pm\sqrt{-1}$ up to a nonzero factor. Therefore, E has four tangency points, two of which are saddle points and two focus points, and E satisfies the genericity conditions (G1) and (G2).

It is easy to check that the unit circle $x^2 + z^2 - 1 = 0$ is the union of two phase curves of C and the points a_1 and a_2. (The phase portrait of C is shown at Figure 2.) Therefore the path $t \to (cost, 0, sint)$, $t \in [0, 2\pi]$ is a smooth closed simple contractible E-path in $S(E)$.

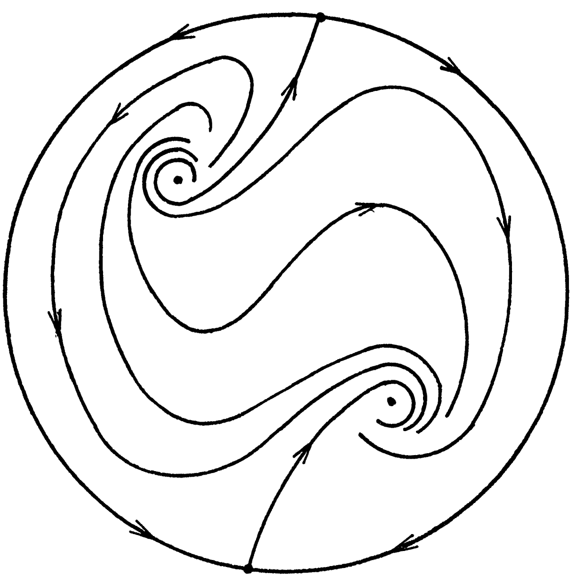

FIGURE 2. The phase portrait of the characteristic vector field (2.5)

CONJECTURE.[5] *Let E be a smooth 2-distribution on M^3 satisfying the genericity conditions (G1) and (G2). Any closed E-path in $S(E) - T(E)$ without return points*[6] *is noncontractible. (Equivalent statement: A characteristic vector field has no contractible cycles.) If it is true, then in the case where $S(E)$ has the topological type of plane or sphere, any closed abnormal path without return points should contain at least one saddle point.*

3. Main theorems

THEOREM 3.1. *An immersed E-path γ defined on an interval $[\alpha, \beta]$ is rigid if and only if the restriction of γ to the open interval (α, β) is a curve in $S(E) - T(E)$ (i.e., all points of the restriction are transversality singular points of E), and each of the end points $\gamma(\alpha)$ and $\gamma(\beta)$ is either a transversality point or saddle point.*

Remark 3.1. The criterion of the local rigidity is exactly the same, and it is a logical corollary of Theorem 3.1.

It follows from the results of the previous section that we can divide all smooth E-paths in $S(E)$ into four types:

Type 1. Paths containing transversality points only. (In particular, closed paths.)

Type 2. Paths $\gamma\colon [\alpha, \beta] \to S$ such that $\gamma(t)$ is a transversality point for all $t \in (\alpha, \beta)$. Exactly one of the end points $\gamma(\alpha)$, $\gamma(\beta)$ is a saddle point, and the other end point is a transversality point.

Type 3. Paths $\gamma\colon [\alpha, \beta] \to S$ such that $\gamma(t)$ is a transversality point for all $t \in (\alpha, \beta)$, and both of the endpoints $\gamma(\alpha)$, $\gamma(\beta)$ are saddle points (the case $\gamma(\alpha) = \gamma(\beta)$ is not excluded).

Type 4. Paths $\gamma\colon [\alpha, \beta] \to S$ such that for at least one $t \in (\alpha, \beta)$the point $\gamma(t)$ is a saddle point.

Using these types of admissible paths in $S(E)$, we can join the results of Section 2 and Theorem 3.1, and obtain the following.

THEOREM 3.2. (1) *Any rigid path is contained in the Martinet surface.*

(2) *Any C^1 E-path in the Martinet surface is of one of the types (1)–(4).*

(3) *Any immersed E-path in the Martinet surface of one of the types (1)–(3) is rigid.*

(4) *Any immersed E-path in the Martinet surface of the type (4) is not rigid.*

Remark 3.2. As we mentioned, the first statement of Theorem 3.2 is proved in [6], [10]. The second statement is proved in our Section 2. So, to prove Theorems 3.1 and 3.2, we have to prove the third and the fourth statements of Theorem 3.2.

[5] This appeared in discussions with A. Agrachev.

[6] A point t_0 is a return point of a pass $\gamma\colon [\alpha, \beta] \to M$ if $t_0 \in (\alpha, \beta)$ and for any $\varepsilon > 0$ there exist $t_1 \in (t_0 - \varepsilon, t_0)$ and $t_2 \in (t_0, t_0 + \varepsilon)$ such that $\gamma(t_1) = \gamma(t_2)$.

Remark 3.3. The local rigidity of paths of the first type is proved in [10], [9], [6]. The global rigidity of non–self-intersecting paths of the type (1) is proved (independently and by a different method) in [2].

Remark 3.4. We have shown that a characteristic vector field might have cycles containing transversality points only. An immersed path in such a cycle might be closed, it might repeat the cycle any finite number of times, and it might be nonclosed, but self-intersecting. By Theorem 3.2, all these paths are rigid as well. In other words, one of the statements of Theorem 3.2 is the rigidity of the restriction to *any* compact interval of *any* periodic solution of the system of ordinary differential equations defined by a characteristic vector field.

Remark 3.5. We have shown that the image of any E-path in S has an immersed parametrization. The third and the fourth statements of Theorem 3.2 concern immersed paths only. Note that a smooth nonimmersed path of the first type is not rigid even if it is simple (without multiple points) and even if the derivative vanishes only at one of the end points. Take, for example, the distribution on $\mathbb{R}^3$ generated by vector fields (1.2) (the Martinet normal form) and the path $t \to (0, t^2, 0)$ defined on $[0, 1]$. It is a smooth simple admissible path in the Martinet surface, it contains no tangency points, and it is easy to show that it is not rigid.

4. Proof of the fourth statement of Theorem 3.2. In this section we prove that if an immersed path γ in S is defined on $[\alpha, \beta]$ and, for some $t_0 \in (\alpha, \beta)$, the point $p = \gamma(t_0)$ is a saddle point, then γ is not rigid. First we reduce this statement to the following.

LEMMA 4.1. *Let E be a distribution generated by vector fields (1.3), and let a be a positive number. Consider a path μ of the form $\mu(t) = (0, 0, t)$ defined on the interval $[-a, a]$. There exists a family μ_ε of C^∞ E-paths defined on $[-a, a]$ such that*

(a) *for any ε, the path μ_ε is not a reparametrization of the path μ;*
(b) *μ_ε tends to μ in the C^1-topology;*
(c) *for any ε, the path μ_ε joins the point $(0, 0, -a)$ to $(0, 0, a)$ (the end points of μ);*
(d) *$\mu_\varepsilon(t) - \mu(t)$ is a flat function[7] at the points $\pm a$.*

Let us show how the fourth statement of Theorem 3.2 follows from Lemma 4.1. Let p be a saddle point, let γ be an immersed E-path in S defined on $[\alpha, \beta]$, and let $\gamma(t_0) = p$, $t_0 \in (\alpha, \beta)$. By Proposition 1.8, there exists a neighbourhood U of p and coordinates (x, y, z) in U such that $p = (0, 0, 0)$, and the restriction of E to U is generated by the vector fields (1.3). By Proposition 2.2, the restriction of γ to a small enough interval containing t_0 is a curve in the intersection of U with either x-axis or z-axis (see Section 1.10). By Proposition 1.8′, we can assume without

[7] A function is called flat at a point if it vanishes at this point along with all its derivatives.

loss of generality that this restriction is a curve in the z-axis. Since γ is immersed, there exists a reparametrization $\tilde{\gamma}$ of γ defined on $[-1, 1]$ such that $\tilde{\gamma}(0) = (0, 0, 0)$, and the restriction of $\tilde{\gamma}$ to a small enough interval $[-\delta, \delta]$ is a path in U of the form $\tilde{\gamma}(t) = (0, 0, t)$. It suffices to prove that the path $\tilde{\gamma}$ is not rigid. Let $a = \delta/2$, and let μ be the restriction of $\tilde{\gamma}$ to the interval $[-a, a]$. Let μ_ε be a family of paths defined on $[-a, a]$ and satisfying the properties formulated in Lemma 4.1. Define a family of E-paths $\tilde{\gamma}_\varepsilon$ on $[-1, 1]$ by the relation $\tilde{\gamma}_\varepsilon(t) = \mu_\varepsilon(t)$ as $t \in [-a, a]$, and $\tilde{\gamma}_\varepsilon(t) = \tilde{\gamma}(t)$ as $t \in [-1, 1] - [-a, a]$. It is clear that none of the curves $\tilde{\gamma}_\varepsilon$ is a reparametrization of $\tilde{\gamma}$, that all these curves join the endpoints of $\tilde{\gamma}$, and that $\tilde{\gamma}_\varepsilon$ tends to $\tilde{\gamma}$ in the C^1-topology. Therefore $\tilde{\gamma}$ is not a rigid path.

Proof of Lemma 4.1. Take any odd function $f(t)$ on the interval $[-a, a]$ which is flat at the points $\pm a$, and a function $g(t)$ satisfying the differential equation

$$g'(t) = \varepsilon f(t)g(t) + \varepsilon^2 t f^2(t) + \varepsilon^3 b t^2 f^3(t)$$

and the initial condition $g(-a) = 0$. Note that the solution $g(t)$ is an even function. (The function $z(t) = g(t) - g(-t)$ satisfies the differential equation $z'(t) = \varepsilon f(t)z(t)$ and the initial condition $z(0) = 0$; therefore $z(t) \equiv 0$.) Therefore $g(a) = 0$. Consider now a family of paths μ_ε: $[-a, a] \to \mathbb{R}^3$ of the form

$$\mu_\varepsilon(t) = (\varepsilon f(t), g(t), t).$$

It is clear that this family satisfies the conditions (a)–(d) of Lemma 4.1. Note also that

$$\mu_\varepsilon'(t) = \varepsilon f'(t) v_1(\mu_\varepsilon(t)) + v_2(\mu_\varepsilon(t)),$$

and therefore the arcs μ_ε are admissible. □

5. Separating surfaces. In this section we introduce the notion of a *separating surface* of an admissible path. The existence of a separating surface is a stronger property than the rigidity. We prove (Lemma 5.1 and Lemma 5.2) the existence of a separating surface of a certain E-path in $S(E)$ where E is a distribution generated by vector fields (1.2) (normal form near a transversality point) or (1.3) (normal form near a saddle point). These results imply the local rigidity of any path of one of the types (1)–(3) given in Section 3.3. They also are the base for the proof of the global rigidity.

5.1. Definition of a separating surface. Let γ be a path defined on an interval $[\alpha, \beta]$, and let $\tilde{\gamma}$ be a path defined on a subinterval $[\tilde{\alpha}, \tilde{\beta}]$ of $[\alpha, \beta]$. Denote by τ the number $\inf\{t \in [\tilde{\alpha}, \tilde{\beta}]: \tilde{\gamma}(t) \notin \operatorname{Im} \gamma\}$. (If $\tilde{\gamma}(t) \in \operatorname{Im} \gamma$ for all $t \in [\tilde{\alpha}, \tilde{\beta}]$, then we define τ to be equal to $\tilde{\beta}$.)

Definition. The restriction of $\tilde{\gamma}$ to $(\tau, \beta]$ will be called the splitting part of $\tilde{\gamma}$ with respect to γ and will be denoted $SP(\tilde{\gamma}, \gamma)$. The point $\gamma(\tau)$ will be called the splitting point, and τ the splitting parameter.

Let γ: $[\alpha, \beta] \to M^3$ be an E-path, and let U be a neighbourhood of the image of γ in M^3. Let G be a surface in U containing the path γ and dividing U onto two open connected parts $U^+$and U^-.

Definition. We will say that G is a separating surface of the path γ in the neighbourhood U if, for any E-path $\tilde{\gamma}$ defined on a subinterval $[\tilde{\alpha}, \tilde{\beta}]$ of $[\alpha, \beta]$ which is C^1–close enough to $\gamma|_{[\tilde{\alpha}, \tilde{\beta}]}$ and starts at a point of Im γ, the curve $SP(\tilde{\gamma}, \gamma)$ is contained in U^+. The set U^+ will be called the positive side of G with respect to γ, and the set U^- the negative side.

It is obvious that the existence of a separating surface of an path γ is a stronger property than the rigidity of γ.

PROPOSITION 5.1. *Assume that an E-path γ has a separating surface in a neighbourhood U of the image of γ. Then γ is a rigid path.*

We need one more property of E-paths that is even stronger than the existence of separating surfaces.

Definition. We will say that G is a strongly separating surface of the path γ in the neighbourhood U if it is a separating surface and if any E-path $\tilde{\gamma}$, defined on a subinterval $[\tilde{\alpha}, \tilde{\beta}]$ of $[\alpha, \beta]$, which is C^1–close enough to $\gamma|_{[\tilde{\alpha}, \tilde{\beta}]}$ and starts at a point of U^+, stays in U^+ for all $t \in [\tilde{\alpha}, \tilde{\beta}]$.

The notions of separating and strongly separating surface are illustrated at Figure 3 and Figure 4 respectively. These notions are well defined for a closed path as well (see Figure 5).

Remark 5.1. It is clear that if G is a separating or strongly separating surface of a path γ defined on $[\alpha, \beta]$, then the same is true for any reparametrization $\hat{\gamma}$ of

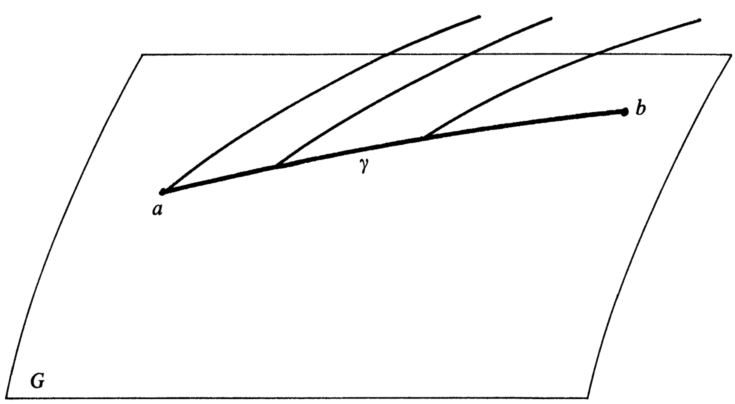

FIGURE 3

A separating surface G of an admissible path γ joining a to b: C^1-close admissible paths starting at a point of γ leave G and stay "above" G as soon as they leave γ.

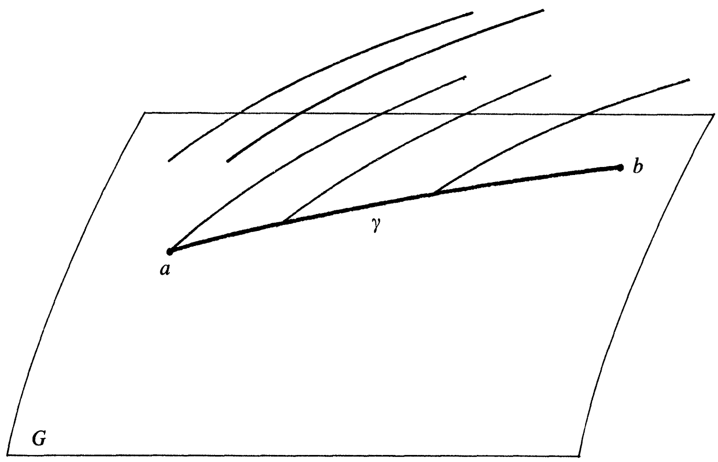

FIGURE 4

A strongly separating surface G of an admissible path γ joining a to b: C^1-close admissible paths starting at a point of γ leave G and stay "above" G as soon as they leave γ. C^1-close admissible paths starting "above" G stay "above" G.

γ preserving the orientation of γ. Moreover, the positive sides of G with respect to γ and $\hat{\gamma}$ are the same. On the other hand, if we change the orientation of γ, then G might be not a separating surface of $\hat{\gamma}$. (See the second example in the next subsection.) In the case of changing the orientation, we preserve the separating surface G, but changing its positive side by the negative one is also possible. (See the first example in the next subsection.)

5.2. Examples. Our first example is the existence of a strongly separating surface of admissible paths in $S(E)$, where E is the distribution generated by vector fields (1.2) (the Martinet normal form). Note that for this distribution any immersed E-path in the Martinet surface $\{x = 0\}$ has the form $x(t) = 0$, $y(t) = t$, $z(t) = 0$, up to reparametrization.

LEMMA 5.1. *Consider a distribution E on $\mathbb{R}^3$ generated by vector fields (1.2) and the E-path $\gamma(t) = (x(t), y(t), z(t)) = (0, t, 0)$ defined on an interval $[\alpha, \beta]$. The surface $G = \{z = 0\}$ is a strongly separating surface in $U = \mathbb{R}^3$ of γ. The positive side of this surface is the semispace $\{(x, y, z)\colon z > 0\}$.*

Proof. Let $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$ be an E-path on $[\tilde{\alpha}, \tilde{\beta}] \subset [\alpha, \beta]$. It follows from the C^1-closeness of $\tilde{\gamma}$ and $\gamma|_{[\tilde{\alpha}, \tilde{\beta}]}$ that

$$\tilde{y}'(t) > 0, \qquad t \in [\tilde{\alpha}, \tilde{\beta}]. \tag{5.1}$$

The admissibility of $\tilde{\gamma}$ implies the existence of functions $u_1(t)$ and $u_2(t)$ such that

$$\dot{\tilde{x}}(t) = u_1(t), \qquad \dot{\tilde{y}}(t) = u_2(t), \qquad \dot{\tilde{z}}(t) = \tilde{x}^2(t)u_2(t). \tag{5.2}$$

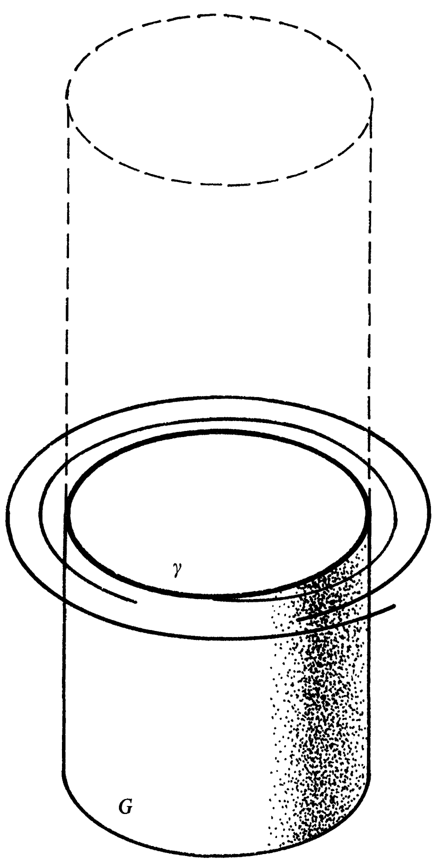

FIGURE 5
The cylinder G is a strongly separating surface of an admissible closed path γ. C^1-close admissible paths starting at a point of γ leave G and stay outside G as soon as they leave γ. C^1-close admissible paths starting outside G stay outside G.

By (5.1), $u_2(t) > 0$ for all t, and therefore the following inequality holds:

$$\tilde{z}'(t) \geqslant 0, \qquad t \in [\tilde{\alpha}, \tilde{\beta}]. \tag{5.3}$$

The relations (5.2) also imply

$$\tilde{z}'(t) = 0 \Leftrightarrow \tilde{x}(t) = 0, \tag{5.4}$$

and the statement of the lemma follows easily from (5.3) and (5.4). □

Remark 5.2. In the same way, we can prove that the path $\gamma(t) = (x(t), y(t), z(t)) = (0, -t, 0)$ has the same separating surface $\{z = 0\}$, but in this case the positive side is the semispace $\{z < 0\}$. Therefore, changing the orientation of γ leads to a change of the positive and negative sides of the separating surface G.

In the second example we consider the distribution E generated by vector fields (1.3) (normal form near a saddle point p), and show the existence of a separating (not strongly separating) surface of any immersed path starting at p.

LEMMA 5.2. *Consider a distribution E on $\mathbb{R}^3$ generated by vector fields (1.3) and the E-path $\gamma(t) = (x(t), y(t), z(t)) = (0, 0, t)$, $t \in [0, \beta]$, $\beta > 0$. The surface $\{y = 0\}$ is a separating surface in $U = \mathbb{R}^3$ of γ. The positive side of this surface is the semispace $\{(x, y, z)\colon y > 0\}$.*

Proof. Let $\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t), \tilde{z}(t))$ be an E-path defined on a subinterval $[\tilde{\alpha}, \tilde{\beta}]$ of $[0, \beta]$, C^1-close enough to the path $\gamma|_{[\tilde{\alpha}, \tilde{\beta}]}$, and such that $\tilde{\gamma}(\tilde{\alpha}) \in \operatorname{Im} \gamma$, i.e., $\tilde{\gamma}(\tilde{\alpha}) = (0, 0, c)$, $c \in [0, \beta]$. The admissibility of $\tilde{\gamma}$ implies the relation

$$\tilde{y}'(t) = (\tilde{x}(t)\tilde{y}(t) + \tilde{x}^2(t)\tilde{z}(t) + b\tilde{x}^3(t)\tilde{z}^2(t))z'(t).$$

Define a function

$$Q(t) = exp\left(-\int_0^t \tilde{x}(s)\tilde{z}'(s)\, ds\right), \qquad t \in [\tilde{\alpha}, \tilde{\beta}].$$

Then

$$(Q(t)\tilde{y}(t))' = Q(t)\tilde{z}'(t)(\tilde{x}^2(t)\tilde{z}(t) + b\tilde{x}^3(t)\tilde{z}(t)).$$

For all $t \in (\tilde{\alpha}, \tilde{\beta}]$ we have

$$Q(t) > 0,$$

$$\tilde{z}'(t) > 0, \tag{5.5}$$

$$\tilde{z}(t) > 0, \tag{5.6}$$

$$\tilde{x}^2(t)\tilde{z}(t) + b\tilde{x}^3(t)\tilde{z}^2(t) > (1/2)\tilde{x}^2(t)\tilde{z}(t) \geqslant 0. \tag{5.7}$$

(The inequalities (5.5)–(5.7) follow from the C^1-closeness of $\tilde{\gamma}$ and γ and from the relation $\tilde{z}(0) \geqslant 0$.) These inequalities imply

$$(Q(t)\tilde{y}(t))' \geqslant 0, \qquad t \in (\tilde{\alpha}, \tilde{\beta}],$$

$$(Q(t)\tilde{y}(t))' = 0 \Leftrightarrow \tilde{x}(t) = 0.$$

Since $\tilde{y}(\tilde{\alpha}) = 0$ and $Q(t) > 0$, we obtain that $\tilde{y}(t) \geqslant 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta}]$ and that if $\tilde{y}(t) = 0$ for some t, then $\tilde{y}(s) = \tilde{x}(s) = 0$ for all $s \leqslant t$, i.e., $\tilde{\gamma}(t)$ is a point of $\operatorname{Im} \gamma$.

This means that $\tilde{y}(t) > 0$ for all $t > \tau$ where τ is the splitting parameter of $\tilde{\gamma}$ with respect to γ. Therefore $\{y = 0\}$ is a separating surface with the positive side $\{y > 0\}$. □

Remark 5.3. The path γ starts at the saddle point $(0, 0, 0)$. One can consider the path $\tilde{\gamma}(t) = (0, 0, -t)$ defined on the interval $[0, a]$ which also starts at the saddle and lies in the same invariant manifold. The same arguments show that $\tilde{\gamma}$ has the same separating surface $\{y = 0\}$, but its positive side is the semispace $\{y < 0\}$. On the other hand, the surface $\{y = 0\}$ is *not* a separating surface for the path $\mu(t) = (0, 0, t)$, $t \in [-a, 0]$, which lies in the same invariant manifold, starts at a transversality point, and ends at the saddle (it follows from Lemma 4.1).

Using Lemma 5.1 and 5.2, the results of Section 1.10 (Proposition 7 and Proposition 1.8′), and the local structure of admissible curves near a saddle point (Proposition 2.2), we obtain the following result: all paths satisfying the condition of Theorem 3.1 are locally rigid (the local rigidity of a path γ on an interval $[\alpha, \beta]$ means the rigidity of its restriction to any small enough interval $[\alpha_1, \beta_1] \subset [\alpha, \beta]$). In the next sections we develop the theory of separating surfaces to prove the global rigidity.

6. Strongly separating surfaces of nonclosed paths in $S(E) - T(E)$. In this section we give an invariant way of constructing a family of strongly separating surfaces of any immersed simple nonclosed admissible path in $S(E) - T(E)$.

The existence of strongly separating surfaces implies the rigidity of any simple nonclosed admissible paths γ containing no tangency points (a part of the first statement of Theorem 3.1.), while the method of their construction will be essentially used in the proofs of the global rigidity of paths of the types (2) and (3). The proof of Theorem 6.1 is based on the choice of special coordinates in a neighbourhood in M^3 of the image of γ. In particular, we prove that the Martinet normal form (1.2) holds not only in a neighbourhood of a transversality point, but also in a neighbourhood of the image of γ—we believe that this result has an independent significance.

Let γ be an immersed simple nonclosed admissible path in the Martinet surface S defined on an interval $[\alpha, \beta]$. Let Γ be an E-curve in S defined on an open interval and such that $\operatorname{Im} \gamma \subset \operatorname{Im} \Gamma$. Take any line subdistribution L of E which is transversal to S at any point of the image of γ. Denote by $G(\Gamma, L)$ the set formed by all leaves of L crossing the image of Γ.

THEOREM 6.1. (1) *There exists a neighbourhood U in M^3 of the image of γ such that $G = U \cap G(\Gamma, L)$ is a strongly separating surface of the path γ in the neighbourhood U.*

(2) *Let U^+ and U^- be the positive and negative sides of G with respect to γ (respectively). Assume that γ starts at a point a and ends at a point b. Let $\tilde{\gamma}$ be a reparametrization of γ starting at b and ending at a (i.e., γ and $\tilde{\gamma}$ have different orientations). Then G is also a strongly separating surface of the path $\tilde{\gamma}$, but the positive side of G with respect to $\tilde{\gamma}$ is U^- and the negative side is U^+.*

COROLLARY. *Any immersed simple nonclosed path in the Martinet surface containing no tangency points is rigid.*

In Section 7 we will prove the same result for *all* immersed admissible paths in $S(E) - T(E)$, including closed and self-intersecting paths. (They might exist if a characteristic vector field has cycles; see section 2.5.) Moreover, we will show that all immersed admissible paths in $S(E) - T(E)$ have a strongly separating surface.

The proof of Theorem 6.1 is based on the choice of a convenient coordinate system in a neighbourhood of the image of γ. Namely, we prove the following.

THEOREM 6.2. *Let L be a 1-dimensional subdistribution of E which is transversal to S at any point of the image of an immersed simple nonclosed admissible path γ in $S(E) - T(E)$. There exists a neighbourhood U in M^3 of the image of γ and a coordinate system X, Y, Z in U such that*

(a) *the restriction of E to U is generated by the vector fields*

$$v_1 = \frac{\partial}{\partial X}, \qquad v_2 = \frac{\partial}{\partial Y} + X^2 \frac{\partial}{\partial Z}, \tag{6.1}$$

(b) *the restriction of L to U is generated by the vector field $\partial/\partial X$,*

(c) *the curve γ has the form $X(t) = 0$, $Y(t) = \pm t$, $Z(t) = 0$ up to a reparametrization preserving the orientation of γ.*

Note that Theorem 6.2 contains the following generalization of the Martinet theorem on local normal form near a transversality singular point:

COROLLARY. *Let γ be a simple nonclosed admissible path in $S(E) - T(E)$. There exists a neighbourhood U of the image of γ in M^3 and coordinates X, Y, Z in U such that the restriction of E to U is generated by the vector fields (6.1).*

Theorem 6.1 follows easily from Theorem 6.2 and Lemma 5.1: We take a neighbourhood U of $\operatorname{Im}\gamma$ and coordinates X, Y, Z in U satisfying the requirements of Theorem 6.2. Then $G(\Gamma, L) \cap U = \{p \in U : Z(p) = 0\}$, and we can use Lemma 5.1. (Remark 5.2 is important to conclude the second statement of Theorem 6.1.)

We complete this section by the proof of Theorem 6.2. We can define a characteristic vector field C in a neighbourhood N in S of the image of γ. If N is small enough, then C has no singular points in N, and since γ is simple and nonclosed, there exist coordinates Y and Z in N such that C has the form $\partial/\partial Y$, and the path γ lies in the set $\{Z = 0\}$. Since the 1-dimensional subdistribution L is transversal to S at the points of the image of γ, it is also transversal to S at all points of N. Therefore there exists a neighbourhood U in M^3 and a function X on U such that $U \cap S = N = \{X = 0\}$, (X, Y, Z) is a coordinate system in U, and the restriction of L to U is generated by the vector field $v_1 = \partial/\partial X$. We can choose a section v_2 of the restriction of E to U, such that $v_2(p) = \partial/\partial Y$ for any $p \in N$ and (v_1, v_2) is a basis of sections of the restriction of E to U. The vector field v_2 has the form $A(X, Y, Z)(\partial/\partial X) + B(X, Y, Z)(\partial/\partial Y) + C(X, Y, Z)(\partial/\partial Z)$, where

$A(0, Y, Z) = C(0, Y, Z) \equiv 0$ and $B(0, Y, Z) = 1$. If U is a small enough neighbourhood (in the direction transversal to S), then the function B does not vanish in U, and therefore the restriction of E to U is generated by vector fields v_1 and $\tilde{v}_2 = \partial/\partial Y + D(X, Y, Z)(\partial/\partial Z)$, where $D = C/B$. The intersection of the Martinet surface S with N is given by the equation $\partial D/\partial X = 0$, and we obtain that $D(0, Y, Z) = (\partial D/\partial X)(0, Y, Z) \equiv 0$. These relations imply that D has the form $X^2 D_1(X, Y, Z)$, where D_1 is a smooth function. The fact that any point of N is a transversality singular point of E implies that the restriction of D_1 to N is a nonvanishing function, and then D_1 is a nonvanishing function if the neighbourhood U is small enough. It follows that the functions $\tilde{X} = X\sqrt{|D_1|}$, Y, Z also form a coordinate system in U. In this coordinate system, the basis of sections $(v_1, \tilde{v}_2)$ has the form (6.1), up to the notation of the coordinates. Any point of the image of γ has zero X- and Z-coordinates, and since γ is immersed it has the form $(0, \pm t, 0)$ up to a reparametrization preserving the orientation.

7. Pasting of separating surfaces. In this section we give a way of pasting of separating surfaces and use this method for the globalization of local results obtained in Section 5. We prove the rigidity of paths of the types (2) and (3), and the rigidity of immersed closed and immersed self-intersecting admissible paths in the Martinet surface.

7.1. Pasting of separating surfaces

THEOREM 7.1 (on pasting of separating surfaces). *Let γ be an E-path defined on $[\alpha, \beta]$. Let $\alpha < t_2 < t_1 < \beta$. Let $\gamma_1 = \gamma|_{[\alpha, t_1]}$, $\gamma_2 = \gamma|_{[t_2, \beta]}$. Assume that the path γ_1 has a separating (resp. strongly separating) surface G_1 in a neighbourhood U_1 of* $\operatorname{Im}\gamma_1$, *and the path γ_2 has a strongly separating surface G_2 in a neighbourhood U_2 of* $\operatorname{Im}\gamma_2$ *such that $G_1 \cap (U_1 \cap U_2) = G_2 \cap (U_1 \cap U_2)$.*

Then $G = G_1 \cup G_2$ is a separating (resp. strongly separating) surface of γ in $U = U_1 \cup U_2$ with the positive side $U^+ = U_1^+ \cup U_2^+$.

Proof. The surface G divides U into two connected open sets. First we prove that one of them is the union of U_1^+ and U_2^+. Taking into account that $U_2^+ \cap G_1 = U_1^+ \cap G_2 = \varnothing$ (it follows easily from the fact that $G_1 \cap (U_1 \cap U_2) = G_2 \cap (U_1 \cap U_2)$), it suffices to prove that $U_1^+ \cap U_2^+ \neq \varnothing$. To prove the latter fact, we take an admissible path $\tilde{\gamma}$ defined on $[t_2, t_1]$ starting at a point of $\operatorname{Im}\gamma$, C^1-close enough to $\gamma|_{[t_2, t_1]}$, and such that $\operatorname{Im}\tilde{\gamma} \not\subset \operatorname{Im}\gamma$. By the properties of the separating surfaces G_1 and G_2, we have

$$SP(\tilde{\gamma}, \gamma_1) \subset U_1^+, \qquad SP(\tilde{\gamma}, \gamma_2) \subset U_2^+.$$

But

$$SP(\tilde{\gamma}, \gamma_1) = SP(\tilde{\gamma}, \gamma_2) = SP(\tilde{\gamma}, \gamma) \neq \varnothing,$$

and therefore U_1^+ and U_2^+ have a nonempty intersection.

We have proved that G divides U into two connected open parts such that the set $U_1^+ \cup U_2^+$ is one of them. Now we will prove that G is a separating surface in U with the positive side $U^+ = U_1^+ \cup U_2^+$. Let $\tilde{\gamma}$ be an E-path defined on a subinterval of $[\alpha, \beta]$ starting at a point of $\operatorname{Im} \gamma$ and C^1-close to γ. Let τ be the splitting parameter of $\tilde{\gamma}$ with respect to γ. If $\tau \geqslant t_2$, then we conclude that

$$SP(\tilde{\gamma}, \gamma) \subset U^+, \tag{7.1}$$

using just the fact that G_2 is a separating surface of γ_2. If the interval of the definition of $\tilde{\gamma}$ is a subinterval of $[\alpha, t_1]$, then we conclude (7.1) using just the fact that G_1 is a separating surface of γ_1. Finally, if $\tilde{\gamma}$ is defined on $[\tilde{\alpha}, \tilde{\beta}]$ where $\tilde{\beta} > t_1$ and $\tau < t_2$, then by the separating property of G_1 we conclude that $\tilde{\gamma}(t_2) \in U_1^+$, and since $\tilde{\gamma}(t_2) \in U_2$ it follows that $\tilde{\gamma}(t_2) \in U_2^+$. Now we use that G_2 is a strongly separating surface of γ_2 and conclude (7.1).

The fact that $U_1^+ \cap (U_1 \cap U_2) = U_2^+ \cap (U_1 \cap U_2)$ (the concordance of the positive sides) also trivially implies that any path that is defined on a subinterval I of $[\alpha, \beta]$, C^1-close to $\gamma|_I$ and that starts at a point of U^+ stays in U^+ for all $t \in I$, provided that both G_1 and G_2 are strongly separating surfaces. Therefore, under this condition, G is a strongly separating surface of γ in U. □

7.2. The rigidity of closed and self-intersecting admissible paths in the Martinet surface. The construction of Section 6 and the theorem on pasting of separating surfaces allow us to conclude that any immersed *closed* or *self-intersecting* path in $S(E) - T(E)$ has a strongly separating surface (see Figure 5), and therefore it is rigid as well as a nonclosed path.

THEOREM 7.2. *Let Γ be any admissible periodic immersed mapping $\mathbb{R} \to S(E) - T(E)$, $I \subset \mathbb{R}$ arbitrary compact interval of the length bigger than or equal to the period of Γ. Denote by γ the restriction of Γ to I. Let L be any line subdistribution of E transversal to the Martinet surface at any point of the image of Γ, and let G be the union of all leaves of L intersecting the image of Γ. There exists a neighbourhood U of the image of Γ in M^3 such that $G \cap U$ is a strongly separating surface of the path γ in U.*

Note that paths satisfying the requirements of Theorem 7.2 might be closed, they might repeat their image, and they might be nonclosed but self-intersecting. Note also that any immersed admissible path in $S(E) - T(E)$ either is simple (has no multiple points) and nonclosed, or satisfies the requirements of Theorem 7.2. Therefore, joining the results of Theorems 6.1 and 7.2, we obtain the following.

THEOREM 7.3. *Any immersed path in $S(E) - T(E)$ has a strongly separating surface.*

As a corollary of this theorem, we obtain a part of Theorem 3.2: any immersed path in $S(E) - T(E)$ is rigid.

Proof of Theorem 7.2. Assume that γ is defined on $[\alpha, \beta]$. It is enough to prove the theorem for the case where $\beta - \alpha$ is equal to the period of the mapping G, i.e., for the case where γ is a simple closed path. (The reduction to this case follows trivially from the definition of a separating surface.) Take any two numbers t_1 and t_2 such that $\alpha < t_2 < t_1 < \beta$, and denote by γ_1 and γ_2 the restrictions of γ to $[\alpha, t_1]$ and $[t_2, \beta]$, respectively. The paths γ_1 and γ_2 satisfy the condition of Theorem 6.1, and we can apply the first statement of this theorem: there exists a neighbourhood U_1 of the image of γ_1 in M^3 such that $G \cap U_1$ is a strongly separating surface of γ_1 in U_1, and there exists a neighbourhood U_2 of the image of γ_2 in M^3 such that $G \cap U_2$ is a strongly separating surface of γ_2 in U_2. The conditions of the theorem 7.1 on pasting of separating surfaces hold, and by this theorem $G \cap (U_1 \cup U_2)$ is a strongly separating surface of γ in $U = U_1 \cup U_2$. □

The construction of a strongly separating surface of a closed simple admissible path γ in S can be effectively described by the blowing-up method (transition to the polar coordinates). Working in a small enough neighbourhood of the image of γ in M^3, we can introduce coordinates $z \in (-\varepsilon, \varepsilon)$, $R \in (1 - \varepsilon, 1 + \varepsilon)$ and $\phi (\text{mod } 2\pi)$ such that the Martinet surface is given by the equation $z = 0$, the image of γ is given by the equations $z = 0$, $R = 1$, and the distribution E is generated by vector fields $\partial/\partial z$ and $\partial/\partial\phi + z^2 A(R, \phi)(\partial/\partial R)$, where A is a periodic function in ϕ. (The proof is similar to that of Theorem 6.2.) By Theorem 7.2, the cylinder given by the equation $R = 1$ is a strongly separating surface of γ (and one can easily conclude that $R > 0$ is its positive side).

7.3. The rigidity of paths of the type (2). Now we will prove that any path γ in the Martinet surface of the type (2) is rigid. The proof is based on the use of Lemma 5.2, Theorem 6.1, and Theorem 7.1.

Without loss of generality, we can assume that γ is defined on $[0, 1]$, that $\gamma(0)$ is a saddle point $p \in S$, and that for all $t \in (0, 1]$ the point $\gamma(t)$ is a transversality singular point.

Let U_1 be a neighbourhood of p and let (x, y, z) be a coordinate system in U_1 such that $p = (0, 0, 0)$ and the restriction of E to U_1 is generated by vector fields (1.3). Take a number $\delta \in (0, 1)$ such that the restriction γ_1 of γ to $[0, 2\delta]$ is a path in U_1 lying in one of the manifolds $\{x = y = 0\}$ or $\{y = z = 0\}$ (see Sections 2.2 and 1.10). By Proposition 1.8′, we can assume without loss of generality that γ_1 lies in the manifold $\{x = y = 0\}$. By Lemma 5.2 the set $G_1 = \{q \in U_1 : y(q) = 0\}$ is a separating surface of γ_1 in U_1. Let L_1 be a line subdistribution of the restriction of E to U_1 generated by the vector field $\partial/\partial x$. This subdistribution is transversal to the Martinet surface $S = \{y + 2xz + 3bx^2z^2 = 0\}$ at any point of $\operatorname{Im} \gamma$ except the point p. Note that the union of all leaves of L_1 crossing $\operatorname{Im} \gamma$ is a subset of G_1.

Denote by γ_2 the restriction of γ to $[\delta, 1]$. The path γ_2 contains no tangency points and therefore there exists a neighbourhood U_2 of $\operatorname{Im} \gamma_2$ and a line subdistribution L_2 of E restricted to U_2 which is transversal to the Martinet surface at any point of the image of γ_2 and such that

$$L_2|_{U_1 \cap U_2} = L_1|_{U_1 \cap U_2}. \tag{7.2}$$

Take a curve Γ_2 defined on an open interval and such that $\operatorname{Im} \gamma_2 \subset \operatorname{Im} \Gamma_2$. The path γ_2 is an immersed simple nonclosed path, and by Theorem 6.1 the union G_2 of all leaves of L_2 crossing $\operatorname{Im} \Gamma_2$ is a strongly separating surface of γ_2 in U_2. (We have to shrink U_2 if necessary.) It follows from (7.2) that $G_2 \cap (U_1 \cap U_2) = G_1 \cap (U_1 \cap U_2)$. We see that the conditions of Theorem 7.1 hold, and by this theorem $G_1 \cup G_2$ is a separating surface of γ in $U_1 \cup U_2$. Therefore γ is rigid.

Remark 7.1. We have proved the existence of a separating surface for any path of the type (2) *starting* at a saddle point. Of course, it implies the rigidity of all paths of the type (2). On the other hand, for paths of the type (2) *ending* at a saddle point, there are no separating surfaces (it follows from Lemma 4.1).

Remark 7.2. The separating surface of γ depends on the subdistribution L (equal to L_1 in U_1 and to L_2 in U_2). It follows from the construction above that the restriction of L to a neighbourhood of any transversality point q of $\operatorname{Im} \gamma$ can be chosen to be an arbitrarily fixed line subdistribution transversal to the Martinet surface. (We have to shrink U_1, if necessary, so that $q \notin U_1$.)

7.4. The rigidity of paths of the type (3). In this section we prove that any immersed path γ of the type (3) is rigid. Without loss of generality, we can assume that γ is defined on $[-1, 1]$, that $\gamma(-1)$ and $\gamma(1)$ are saddle points, and that for all $t \in (0, 1)$ the point $\gamma(t)$ is a transversality singular point.

Let γ_1 be the restriction of γ to $[-1, 1/2]$. Consider also the path γ_2, which is the restriction of $-\gamma$ to the same interval. ($-\gamma$ is a path defined on $[-1, 1]$ by the relation $(-\gamma)(t) = \gamma(-t)$).

Both γ_1 and γ_2 are paths of the type (2) *starting* at a saddle point. In Section 7.3 we showed that there are neighbourhoods U_1 and U_2 of the images of γ_1 and γ_2 and separating surfaces $G_1 \subset U_1$ and $G_2 \subset U_2$ of γ_1 and γ_2 respectively. The images of γ_1 and γ_2 intersect, and therefore $U = U_1 \cap U_2$ is some neighbourhood of $\gamma(0)$.

It follows from the Remark 7.2 that we can take U_1, U_2, G_1, and G_2 such that $G_1 \cap W = G_2 \cap W = G$, where $W = U_1 \cap U_2$. Let us show that

$$(U_1^+ \cap W) \cap (U_2^+ \cap W) = \varnothing. \tag{7.3}$$

To prove it, consider the path $\gamma_\delta = \gamma|_{[-\delta, \delta]}$, where δ is chosen such that γ_δ is contained in W. Then G is a separating surface of γ_δ in W. The separating property of G_1 implies that the positive side of G with respect to γ is the set $U_1^+ \cap W$. Similarly, G is a separating surface of $-\gamma_\delta$ in W, and the positive side of G with respect to $-\gamma_\delta$ is the set $U_2^+ \cap W$. On the other hand, we can use the second statement of Theorem 6.1 and conclude that the positive side of G with respect to $-\gamma_\delta$ is the set $U_1^- \cap W$, and (7.3) follows.

Now we prove that any E-path $\tilde{\gamma}$ defined on $[-1, 1]$, connecting p_1 and p_2 and sufficiently C^1-close to γ, lies in the image of γ (and then the rigidity of γ follows).

Consider the point $\tilde{\gamma}(0)$. If $\tilde{\gamma}(0) \in \operatorname{Im} \gamma$, then from the separating property of G_1 it follows that $\tilde{\gamma}|_{[-1,0]} \subset \operatorname{Im} \gamma$, and it follows from the separating property of G_2 that $(-\tilde{\gamma})|_{[-1,0]} \subset \operatorname{Im} \gamma$, whence $\operatorname{Im} \tilde{\gamma} \subset \operatorname{Im} \gamma$.

It remains to show that the case $\tilde{\gamma}(0) \notin \operatorname{Im} \gamma$ is impossible. Assume that $\tilde{\gamma}(0) \in \operatorname{Im} \gamma$. Then the separating property of G_1 implies $\tilde{\gamma}(0) \in U_1^+ \cap W$, and the separating property of G_2 implies $\tilde{\gamma}(0) \in U_2^+ \cap W$, which contradicts (7.3).

Remark 7.3. In this proof, we have not used that the end points of γ are different, and therefore the proof works for saddle connections as well.

We have completed the proof of Theorems 3.1 and 3.2.

References

[1] A. A. Agrachev and A. V. Sarychev, *Abnormal sub-Riemannian geodesics: Morse index and rigidity*, preprint, 1993.

[2] ———, *Strong minimality of abnormal geodesics for 2-distributions*, to appear in J. Dynam. Contr. Syst.

[3] V. I. Arnold and Yu. S. Ilyashenko, "Ordinary Differential Equations" in *Dynamical Systems I*, Encyclopedia of Mathematical Sciences **1**, Springer-Verlag, Berlin, 1988.

[4] V.I. Arnold, A. N. Varchenko, and S. M. Gusein-Zade, *Singularities of Differentiable Maps, Vol. 1*, Birkhauser, Boston, 1985.

[5] R. Bryant and L. Hsu, *Rigidity of integral curves of rank two distributions*, Invent. Math. **114** (1993), 435–461.

[6] W.-S. Liu and H. J. Sussmann, *Shortest paths for sub-Riemannian metrics on rank-two distributions*, to appear in Trans. Amer. Math. Soc.

[7] J. Martinet, *Sur les singularites des formes differentialles*, Ann. Inst. Fourier (Grenoble) **20** (1970), 95–178.

[8] R. Montgomery, *Geodesics which do not satisfy geodesic equations*, preprint, 1991.

[9] ———, *On sub-Riemannian geodesics*, preprint, 1992.

[10] ———, *Abnormal minimizers*, to appear in SIAM J. Control Optim.

[11] ———, *A survey of singular curves in sub-Riemannian geometry*, preprint, 1994.

[12] M. Zhitomirskii, *Rigid and abnormal line subdistributions of 2-distributions*, to appear in J. Dynam. Contr. Syst.

[13] ———, *Degenerations of differential forms and Pfaffian structures*, Russian Math. Surveys **46**: 5 (1991), 53–90.

[14] ———, *Typical Singularities of Differential 1-forms and Pfaffian Equations*, Transl. Math. Monographs **113**, Amer. Math. Soc., Providence, 1992.

Department of Mathematics, Technion-Israel Institute of Technology, 3200 Haifa, Israel