


Chapter 1 Polynomials and Modeling

1.1 Linear Functions

Recall that a line is a function of the form $y = mx + b$, where m is the slope of the line (how steep the line is) and b gives the y -intercept (where the line crosses the y -axis). Using the notation given in Chapter 0, a line is a **linear function** in the form $f(x) = mx + b$.

 **Example 1.1** Find the intercepts and slope of the linear function $f(x) = \frac{3}{4}x - \frac{15}{4}$.

 **Solution** To find the y -intercept, you set $x = 0$. Therefore, we have

$$f(0) = \frac{3}{4}(0) - \frac{15}{4} = -\frac{15}{4}$$

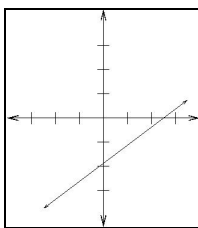
Thus we say that the y -intercept is the ordered pair $(0, -\frac{15}{4})$. Set $y = f(x) = 0$ to find the x -intercept,

$$\begin{aligned} f(x) &= \frac{3}{4}x - \frac{15}{4} = 0 \\ 3x - 15 &= 0 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

So, the x -intercept is the ordered pair $(5, 0)$. ◇

In this form the slope is the constant in front of the x and, therefore is equal to $\frac{3}{4}$. The intercepts of this function can be seen in Figure 1.1.

 **Figure 1.1** Graph of Example 1.1



Example 1.2 Given $f(x) = -\frac{2}{5}x + 3$, if x increases by 10, what is the corresponding change in $f(x)$?

Solution The slope of a line is the amount of change in y when x increases by one unit. Since $m = \frac{\Delta y}{\Delta x} = -\frac{2}{5} = -0.4$, therefore, if x increases by 1, y decreases by 0.4 units.

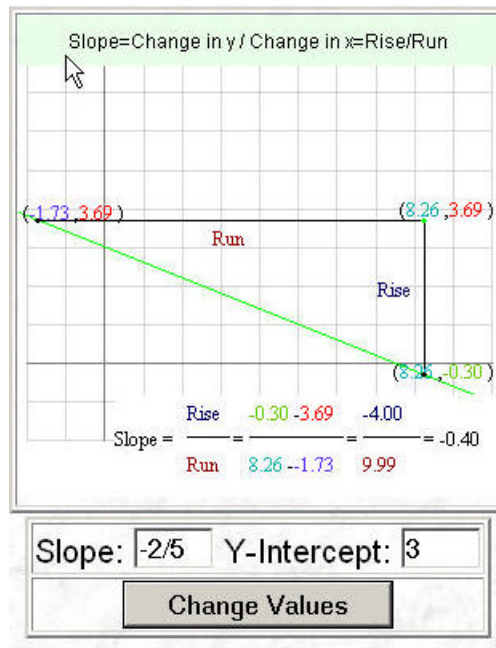
Now, we have $x = 10$, then

$$\begin{aligned} \Delta y &= m \cdot \Delta x \\ &= -0.4 \cdot 10 \\ &= -4 \end{aligned}$$

◇

Using the Lines and Slope Applet, we adjust the change in x (the “Run”) to be about 10 and look for the corresponding change in the function value (the “Rise”). From the screen we see this to be -4 .

Figure 1.2 Applet for Example 1.2



BUSINESS APPLICATIONS


The aspects of linear functions we just explored have particular meaning when discussing real life applications of linear functions. One application of linear functions can be seen in business when discussing linear cost, revenue, and profit.

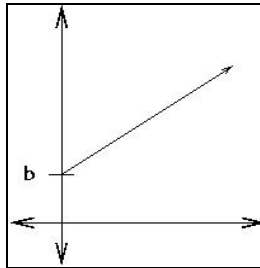
A **linear cost function** consists of two parts: variable costs (costs that depend on the number of items produced) and fixed costs (costs independent of the number of items produced). The total cost function is the sum of these costs.

Total Costs = Variable Costs + Fixed Costs

$$C(x) = mx + b$$

where x represents the number of items produced, m represents the cost to produce each item, and b represents the fixed costs.


 **Figure 1.3** Graph of a Generic Cost Function

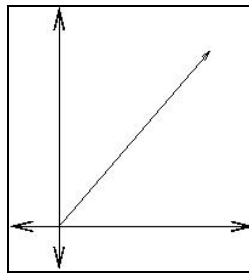


A **linear revenue function** is found by multiplying the selling price of each item sold, p , by the total number of items sold, x . Thus, we have

Revenue = (Selling Price)(Quantity)

$$R(x) = px$$


 **Figure 1.4** Graph of a Generic Revenue Function

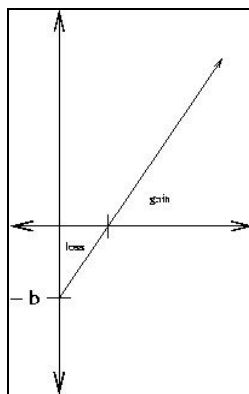


Finally, a **linear profit function** represents the difference between the amount of money a company gains through revenue and the amount of money it pays out in the form of costs.


$$\begin{aligned} P(x) &= R(x) - C(x) = px - (mx + b) = (px - mx) - b \\ &= (p - m)x - b \end{aligned}$$

In this final format, notice that $p - m$ represents the profit for each item made and sold.

 **Figure 1.5** Graph of a Generic Profit Function



From Figure 1.5, you can see that when $P(x) = 0$, a company is not losing or gaining money. This point is called the **break-even point**.

 **Example 1.3** A company is manufacturing and selling insulated mugs. The company has monthly fixed costs of \$1500 and there is a total monthly cost of \$1800 when producing 100 mugs. Each mug sells for \$7.

- Find the cost, revenue, and profit functions for the mug manufacturer, assuming each is a linear function.
- How many mugs must the company make and sell in order to break even?

 **Solution**

a. First, gather all of the cost information together in ordered pairs of the form $(x, C(x))$. Thus, we have $(0, 1500)$ and $(100, 1800)$. Next, find the slope between the two points:

$$m = \frac{1800 - 1500}{100 - 0} = \frac{300}{100} = 3$$

Since a linear cost function has the form $C(x) = mx + b$, and we know the values of m and b (b is the fixed cost), the cost function is $C(x) = 3x + 1500$.

To find the revenue function, all we need is the selling price, which is given as \$7. Therefore, the revenue function is $R(x) = 7x$.


Finally, the profit function can be found using the cost and revenue functions we just found, as follows:


$$P(x) = R(x) - C(x) = 7x - (3x + 1500) = 4x - 1500$$

b. The company will break even when $P(x) = 0$ (the x -intercept of the profit function). Therefore,

$$\begin{aligned} P(x) &= 4x - 1500 = 0 \\ 4x &= 1500 \\ x &= \frac{1500}{4} = 375 \end{aligned}$$

So, by making and selling 375 mugs, the company will neither gain nor lose money. \diamond

 **Example 1.4** A company has costs given by $C(x) = 55.5x + 1000$ and revenue given by $R(x) = 80x$, where x represents the number of items the company makes and sells. Find the number of items the company needs to produce in order to break even.

 **Solution** A company breaks even when $P(x) = 0$. Therefore, finding profit,

$$P(x) = R(x) - C(x) = 80x - (55.5x + 1000) = 24.5x - 1000$$

and setting it equal to zero we have

$$\begin{aligned} P(x) &= 24.5x - 1000 = 0 \\ x &= \frac{1000}{24.5} \approx 40.82 \end{aligned}$$


Thus, if it were possible to produce and sell a fraction of an item then they would break even at $\frac{2000}{49}$ items. However, they most likely cannot produce and sell fractional items so this means that the company will never truly break even. (If they sell 40 items they will lose a slight amount of money, but if they sell 41 items they will make a slight amount of money.) \diamond

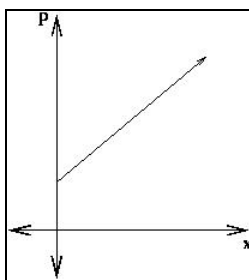
ECONOMIC APPLICATIONS


Another application of linear functions can be seen in the discussion of supply and demand. A **linear supply function** tells us the price at which a producer is willing to supply exactly x units of a product to the marketplace.

$$S(x) = p = mx + b$$

Generally, because producers are trying to make money, as the price they are given for each product increases, they are willing to increase the amount of product they supply to the marketplace. Thus, as x increases, so does p (or $S(x)$), as shown in Figure 1.6.

 **Figure 1.6** Graph of a Generic Supply Function



 **Example 1.5** Suppliers are willing to provide 20 light bulbs at a price of 30 cents a piece and are willing to supply 50 light bulbs for 60 cents each. Find the linear supply function describing the light bulb market.

✓ **Solution** Since the supply function is a linear function of the form $S(x) = mx + b$, we can organize the given information as ordered pairs of the form $(x, S(x))$. Thus, we have $(20, 0.30)$ and $(50, 0.60)$ as two points on the line. Now we can find the slope between the two points

$$m = \frac{0.60 - 0.30}{50 - 20} = \frac{-0.30}{30} = \frac{1}{100}$$

and use the point-slope formula of a line to find the supply function as follows:

$$S(x) - 0.60 = \frac{1}{100}(x - 20)$$

$$S(x) = \frac{1}{100}x + \frac{2}{5}$$

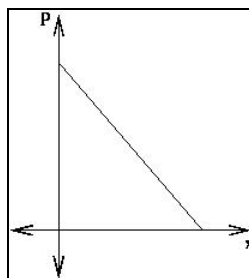
◇

A **linear demand function** tells us the price at which a consumer is willing to buy exactly x units of a product from the marketplace.

$$D(x) = p = mx + b$$

Generally, because consumers are trying to save money, as the price of a product increases, they tend to buy less of that product. Thus, as x increases, p (or $D(x)$) decreases, as shown in Figure 1.7.

□ **Figure 1.7** Graph of a Generic Demand Function



From Figure 1.6 it is easy to see that the lowest price producers are willing to accept for their product is the y -intercept of the linear supply function. Likewise, from Figure 1.7, you can see that the highest price consumers are willing to pay for a product is the y -intercept of the linear demand function.

↙ **Example 1.6** Consumers are willing to buy 70 light bulbs at a price of 50 cents a piece and are willing to buy 30 light bulbs for 70 cents each. What is the highest price consumers are willing to pay for a light bulb, assuming a linear demand function?

✓ **Solution** Since the demand function is a linear function of the form $D(x) = mx + b$, we can organize the given information as ordered pairs of the form $(x, D(x))$. Thus, we have $(70, 0.50)$ and $(30, 0.70)$ as two points on the line. Now we can find the slope between the two points

$$m = \frac{0.50 - 0.70}{70 - 30} = \frac{-0.20}{40} = \frac{-1}{200}$$

and use the point-slope formula of a line to find the demand function as follows:

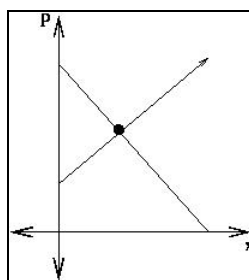
$$D(x) - 0.70 = \frac{-1}{200}(x - 30)$$


$$D(x) = \frac{-1}{200}x + \frac{17}{20}$$


Therefore, the highest price consumers are willing to spend for a light bulb is $\$ \frac{17}{20} = 0.85$. \diamond

If you overlay the supply and demand functions, you can find the point where they intersect, called the **equilibrium point**, demonstrating the so called Law of Supply and Demand.

 **Figure 1.8** Graph of Market Equilibrium



 **Example 1.7** Using the supply and demand functions found for the light bulb market in the previous two examples, find the equilibrium point for the light bulb market.

 **Solution** The equilibrium point is found by finding the intersection point of the supply and demand functions of the market. We found the supply function to be $S(x) = 0.01x + 0.1$ and the demand function to be $D(x) = -0.005x + 0.85$. By setting these equal to each other, we can find the equilibrium quantity, x , as follows:

$$-0.005x + 0.85 = 0.01x + 0.1$$

$$0.75 = 0.015x$$

$$50 = x.$$

By plugging this value into either the supply or demand function, we can then find the equilibrium price.

$$S(x) = 0.01(50) + 0.1 = 0.60 = p$$


Thus, the equilibrium point for the light bulb market is $(50, 0.60)$, meaning that both consumers and producers will be happy with buying and selling 50 light bulbs at a price of 60 cents a piece. \diamond

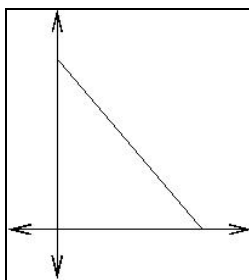
INVESTMENT APPLICATION

A final application of linear functions can be seen when talking about the depreciation of an object over time. An object depreciates if it loses value over time. If the object loses value at a constant rate, then it is said to be depreciating linearly. A **linear depreciation function** is given by


$$V(t) = mt + b$$

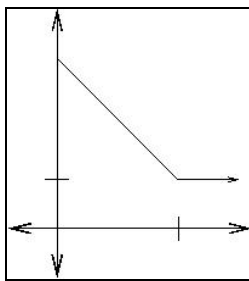
where t is the amount of time over which the object is losing value, m is the rate of depreciation (value lost per time period), and b is the initial value of the object. Notice that m should always be negative since the value is decreasing over time.


 **Figure 1.9** Graph of the Value of an Object Over Time




The lowest possible value for an object is \$0. However, it is possible for an object to retain some amount of value, indefinitely. The **scrap value** of an object is the lowest value an object obtains.

 **Figure 1.10** Non-Zero Scrap Value of an Object



 **Example 1.8** Patti buys a car for \$17,575. The value of her car linearly depreciates to \$5400 over 10 years. Find the value of Patti's car as a linear function of the number of years since she bought the car.

 **Solution** The value of the car is a linear function of the form $V(t) = mt + b$, where b represents the value of the car when $t = 0$ (the original value of the car). Therefore, the value of Patti's car is of the form $V(t) = mt + 17575$. To find the value of m , use the value of the car when $t = 10$:

$$\begin{aligned} V(10) &= m(10) + 17575 = 5400 \\ 10m &= -12175 \\ m &= -1217.5 \end{aligned}$$


Thus, the value of Patti's car is given by $V(t) = -1217.5t + 17575$ when $0 \leq t \leq 10$. ◇

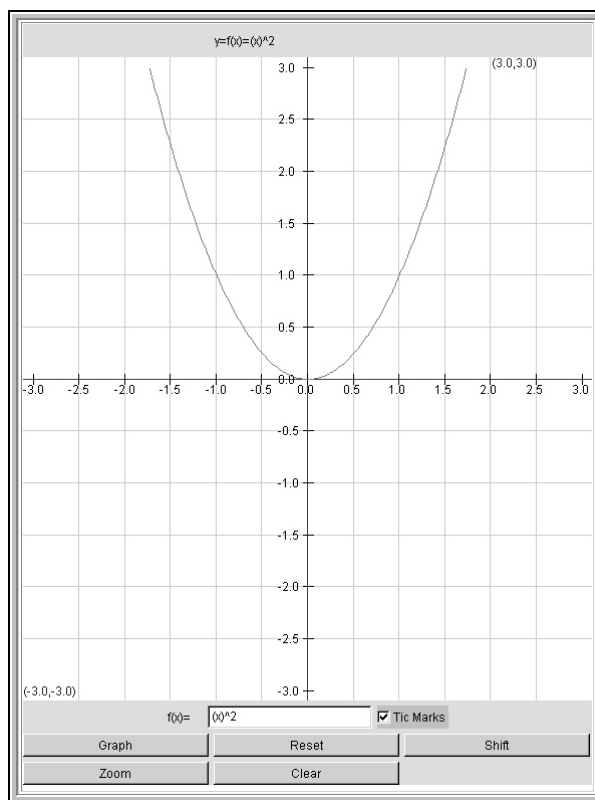
1.2 Quadratic Functions

While linear data always increases or always decreases at a constant rate, not all data behaves in such a manner. To account for more complicated situations, we will begin to introduce more complex functions. The first of these functions will be quadratic functions.

A **quadratic function** is a function of the form $f(x) = ax^2 + bx + c$, where a , b , and c are real numbers and $a \neq 0$.

The simplest quadratic function is $f(x) = x^2$ ($a = 1, b = 0, c = 0$). Its graph can be found by using the Plotting Applet and is shown in Figure 1.11.

 **Figure 1.11** Graph of $f(x) = x^2$ using the Plotting Applet



The graph of $f(x) = x^2$, and every quadratic function, is known as a **parabola**. Every parabola has a lowest (or highest) point which is known as the **vertex** of the parabola. From Figure 1.11 you can see that the vertex of the graph of $f(x) = x^2$ is at $(0, 0)$. You can also see from Figure 1.11 that if the graph was folded along a vertical line through the vertex ($x = 0$), the two halves of the parabola would lie exactly on top of one another. This demonstrates the property that all quadratic functions are **symmetric** with respect to the vertical line through the vertex, which is known as the **axis** of the parabola.



Example 1.9 Graph the following quadratic functions on the same axes. Studying how does the value a effect the value of a quadratic function.

a. $f(x) = x^2$

c. $f(x) = 2x^2$

e. $f(x) = -2x^2$

b. $f(x) = \frac{1}{2}x^2$

d. $f(x) = -\frac{1}{2}x^2$

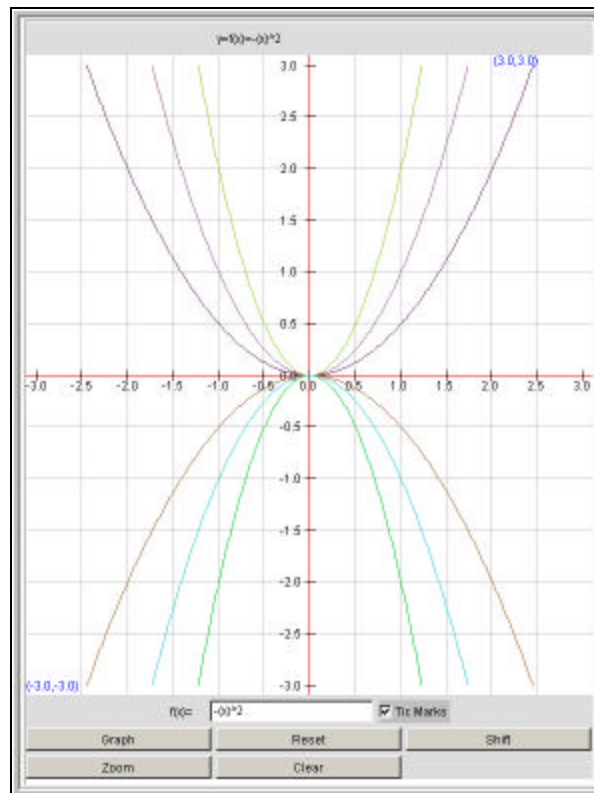
f. $f(x) = -x^2$



Solution Using the Plotting Applet we can graph all six functions, shown in Figure 1.12.




Figure 1.12 Graphs for Example 1.9 using the Plotting Applet






Notice that the vertex of each of these parabolas is at $(0, 0)$ and the axis of symmetry of each parabola is $x = 0$. ◇

Example 1.9 shows the effect of *only* changing the value of a in $ax^2 + bx + c$. Notice that when $a > 0$, the parabola opens upward and the vertex is the lowest point on the graph, while when $a < 0$, the parabola opens downward and is the vertex is the highest point on the graph. By comparing the graph of F to A, D to B, and E to C, we can see that in multiplying $f(x)$ by -1 , the graph of $f(x)$ is reflected

over the x -axis, which is known as a **vertical reflection** of $f(x)$. By comparing the graphs of B and C to A and the graphs of E and D to F, we can see that the magnitude of a determines how fast the graph of $f(x)$ is increasing or decreasing. When $0 < |a| < 1$, we say the graph of $f(x)$ is **vertically compressed** by a factor of a and when $|a| > 1$, we say the graph of $f(x)$ is **vertically expanded** by a factor of a . Finally, when $|a| = 1$, we have our simplest parabola, or its reflection across the x -axis, which all other parabolas are compared to. These results are summarized in Table 1.1.

 **EXAMPLE 1.10** Without graphing, how will the graph of $g(x) = -3x^2$ be different from the graph of $f(x) = x^2$?


 **Solution** The only difference between the two functions is the value of a . Since $a = -3$, the graph of $g(x)$ can be found by vertically reflecting $f(x)$ and then vertically expanding it by a factor of 3. 

 **EXAMPLE 1.11** Graph the following quadratic functions on the same axes and find the vertex of each.

a. $f(x) = x^2$

b. $g(x) = x^2 + 2$

c. $h(x) = x^2 - 3$

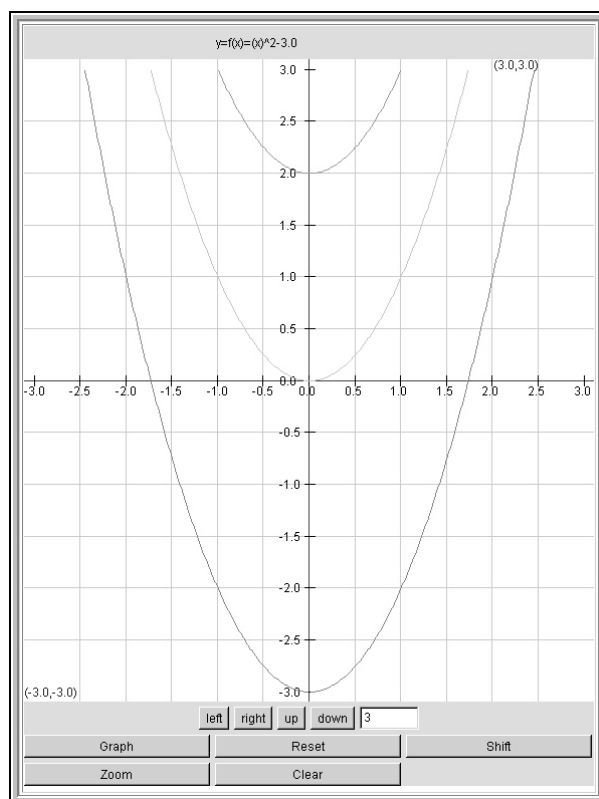
 **Solution** Using the Plotting Applet, we can graph all three functions as shown in Figure 1.13. By examination we see that

a. The vertex of $f(x)$ is at $(0, 0)$.

b. The vertex of $g(x)$ is at $(0, 2)$.

c. The vertex of $h(x)$ is at $(0, -3)$.

Figure 1.13 Graphs for Example 1.11 using the Plotting Applet



Example 1.11 shows the effect of *only* changing the value of c in $ax^2 + bx + c$. If c is increased then the parabola will move up and if c is decreased then the parabola will move down. This type of movement is known as a **vertical shift**. These properties are summarized in Table 1.1.

EXAMPLE 1.12 Without graphing, how will the graph of $g(x) = x^2 + 2x + 5$ be different from the graph of $f(x) = x^2 + 2x + 1$?

Solution The only difference between the two functions is the value of c . Since the value of c is increased by 4, the graph of $g(x)$ can be found by vertically shifting the graph of $f(x)$ up 4 units. In order to compare two quadratic functions where the values of a , b , and/or c change, it is easier to look at the **standard form** of a quadratic function. By completing the square we get

$$\begin{aligned} f(x) &= ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \left(c - \frac{b^2}{4a}\right) \\ &= a(x - h)^2 + k \end{aligned}$$

where $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a} = f(h)$. In standard form, the graph of the parabola can be found by shifting the graph of $f(x) = x^2$ horizontally (right if $h > 0$, left if $h < 0$) by h units, vertically expanding or contracting by a factor of a , vertically reflecting if $a < 0$, and then shifting vertically by k units. Notice that by performing this transformation, the graph of a quadratic in standard form has a vertex at the point (h, k) .



Table 1.1 Summary Chart for Parabolas

Let $f(x) = a(x-h)^2 + k$. Comparing this to the parabola x^2 ,

1. If $a > 0$ then $f(x)$ opens upward and the vertex is the lowest point.
2. If $a < 0$ then $f(x)$ vertically reflected and the vertex is the highest point.
3. If $0 < |a| < 1$ then $f(x)$ is vertically compressed relative to x^2 .
4. If $|a| > 1$ then $f(x)$ is vertically expanded relative to x^2 .
5. If $k > 0$, then $f(x)$ is vertically shifted upwards by k units.
6. If $k < 0$, then $f(x)$ is vertically shifted downwards by k units.
7. If $h > 0$, then $f(x)$ is horizontally shifted to the right by h units.
8. If $h < 0$, then $f(x)$ is horizontally shifted to the left by h units.
9. The vertex is at (h, k) .

Example 1.13 Find the vertex of the parabola $f(x) = 2x^2 - 3x + 6$ and determine whether it is a maximum or minimum.

Solution The x -coordinate of the vertex is given by

$$x = \frac{-b}{2a} = \frac{-(-3)}{2(2)} = 0.75$$

The y -coordinate of the vertex can be found by plugging in the corresponding x value:

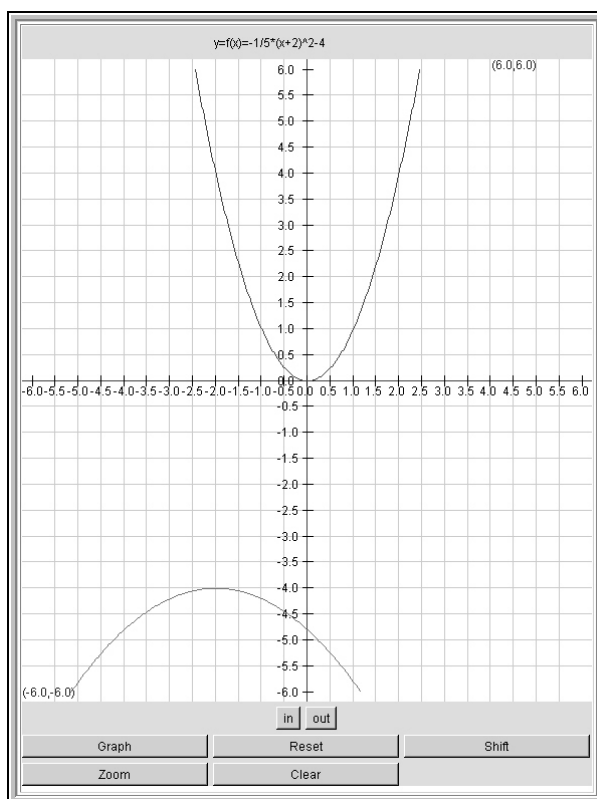
$$f(0.75) = 2(0.75)^2 - 3(0.75) + 6 = 4.875$$

Therefore, the vertex of the parabola is located at $(0.75, 4.875)$. Since $a = 2 > 0$, then the parabola opens upward from the vertex, so the vertex is a minimum. \diamond

Example 1.14 How is the graph of $f(x) = \frac{1}{5}(x+2)^2 - 4$ different from the graph of $g(x) = x^2$?

Solution The graph will be horizontally shifted to the left by 2 units, reflected over the x -axis, vertically contracted by a factor of $\frac{1}{5}$, and then vertically shifted down 4 units. Using the Plotting Applet, these transformations can be seen in Figure 1.14.

Figure 1.13 Graphs for Example 1.11 using the Plotting Applet



Many real-life situations are quadratic in nature, as they portray information that increases to a point and then decreases, such as revenue or profit, or decreases to a point and then increases, such as costs.

Example 1.15 Given the demand of watches to be $p = -3x + 60$, how many watches must be sold in order to maximize revenue?

Solution Recall that revenue is given by $R(x) = px$ where p = price per item sold and x = number of items sold. When price is not fixed, but determined by the buying habits of consumers, price is given as the demand function. Thus,

$$R(x) = px = (-3x + 60)x = -3x^2 + 60x$$

Notice that the revenue function is a quadratic function. Because $a = -3 < 0$, the revenue function will be a parabola that opens downward when it is graphed. Thus, the vertex is the maximum point on the revenue function. The x coordinate of the vertex is given by

$$x = \frac{-b}{2a} = \frac{-60}{2(-3)} = 10$$

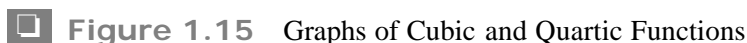
Therefore, 10 watches must be sold in order to maximize revenue.

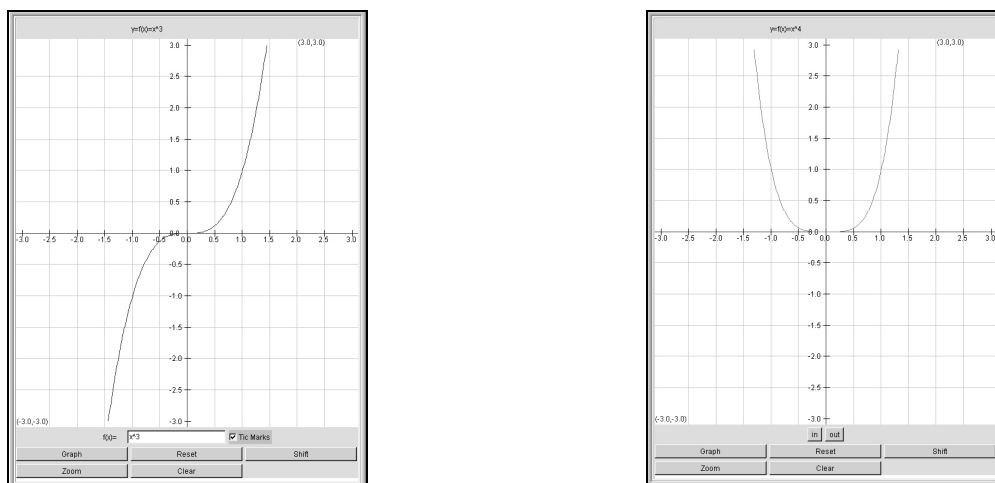


1.3 Cubic and Higher Polynomials

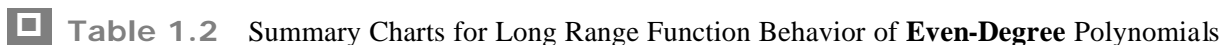
Linear and quadratic functions are two functions belonging to a family of functions called **polynomials**. In general, a polynomial is a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ where a_0, a_1, \dots, a_n are real numbers and n is a whole number, giving the **degree** of the polynomial. With this definition, linear functions ($f(x) = ax + b$) are known as first-degree polynomials, while quadratic functions ($f(x) = ax^2 + bx + c$) are known as second-degree polynomials.

Two higher degree polynomials that will be encountered often are third-degree polynomials (also known as **cubic functions**) and fourth-degree polynomials (also known as **quartic functions**). Their most basic graphs are shown in Figure 1.12.

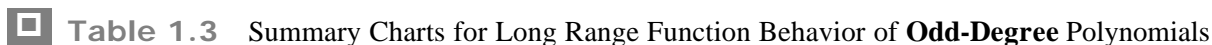
 **Figure 1.15** Graphs of Cubic and Quartic Functions



To determine how polynomial functions behave when their x values become negatively and positively large (as x approaches negative and positive infinity), without looking at their graphs (since they can be hard to graph without the help of technology), it is only necessary to look at the leading term, $a_n x^n$. Tables 1.2 and 1.3 list the long range behavior of polynomials.

 **Table 1.2** Summary Charts for Long Range Function Behavior of **Even-Degree** Polynomials

Leading Coefficient	As $x \rightarrow -\infty$	As $x \rightarrow \infty$
$a_n > 0$	$f(x) \rightarrow \infty$	$f(x) \rightarrow \infty$
$a_n < 0$	$f(x) \rightarrow -\infty$	$f(x) \rightarrow -\infty$

 **Table 1.3** Summary Charts for Long Range Function Behavior of **Odd-Degree** Polynomials

Leading Coefficient	As $x \rightarrow -\infty$	As $x \rightarrow \infty$

$a_n > 0$	$f(x) \rightarrow -\infty$	$f(x) \rightarrow \infty$
$a_n < 0$	$f(x) \rightarrow \infty$	$f(x) \rightarrow -\infty$

Example 1.16 Describe the end behavior of the following functions

a. $f(x) = 2x^5 - 4x^4 + 3x^2 - 6$

b. $g(x) = -3x^4 + 6x - 9$

c. $h(x) = -4x^7 - 8x^2 + 2$

Solution

a. $f(x)$ is an odd-degree polynomial since $n=5$, and because $a_n=2>0$, we know as $x \rightarrow \infty$, $f(x) \rightarrow \infty$ and we know as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.

b. $g(x)$ is an even-degree polynomial since $n=4$, and because $a_n=-3<0$, we know as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$ and we know as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.

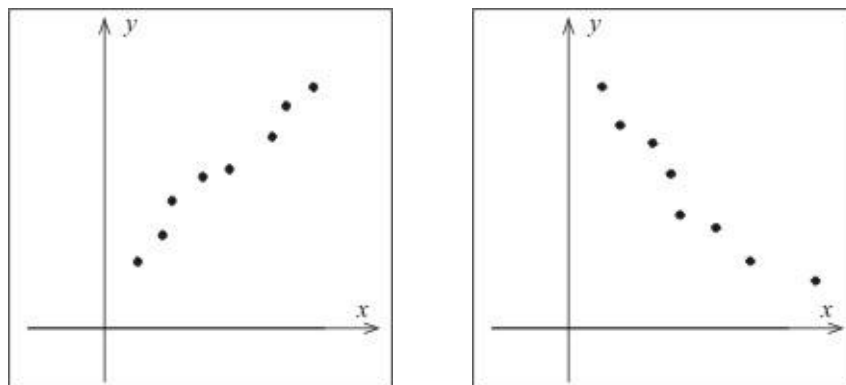
c. $h(x)$ is an odd-degree polynomial since $n=7$, and because $a_n=-4<0$, we know as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$ and we know as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. \diamond

1.4 Modeling with Polynomials

In certain situations, data is collected and used to describe a phenomenon that is occurring. In order to describe the situation accurately and be able to make predictions of future occurrences, a mathematical model is needed. In this section we will discuss the use of polynomial models.

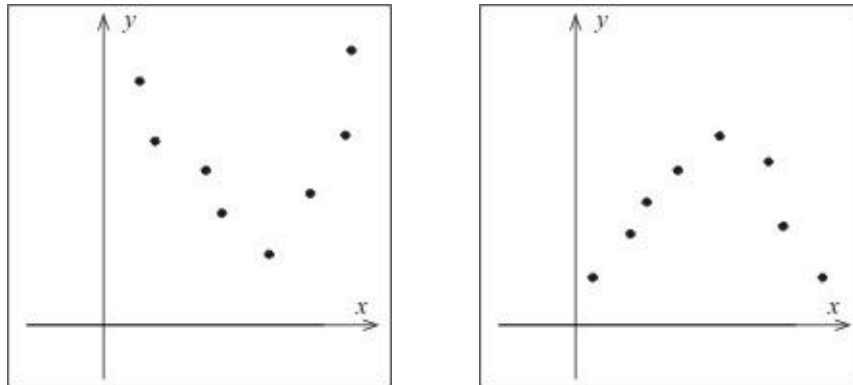
When thinking of using a linear model, the data should fall, more or less, in a straight line and be always increasing or always decreasing. Figure 1.16 shows examples of linear data sets.

Figure 1.16 Graphs of Linear Data



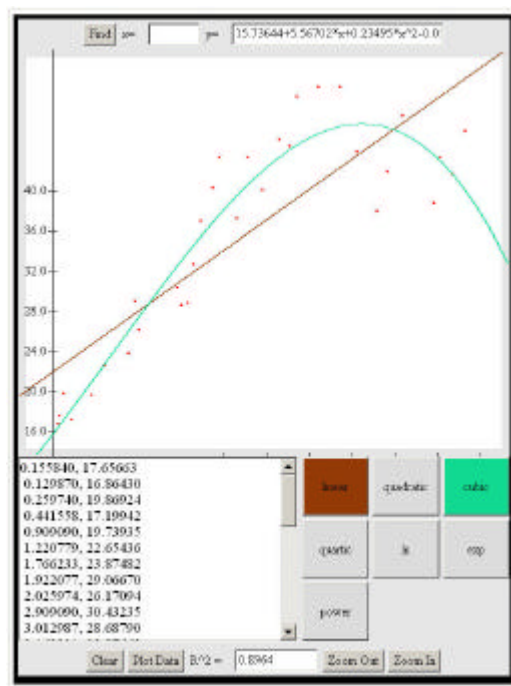
When data increases and then decreases or decreases and then increases, a quadratic model should be explored. Figure 1.17 shows quadratic data.

Figure 1.17 Graphs of Quadratic Data



If the behavior is more complex, even higher order polynomials can be used to fit the data. In order to find a model that explains a set of given data, you can use the Modeling applet.

Figure 1.18 Modeling Applet



Notice that in addition to giving you the equation of the model at the top of the applet, you are also given an R^2 value at the bottom of the applet. This value is the square of the correlation coefficient. The **correlation coefficient** tells you how well your model explains the data and thus, how accurate it would be at predicting values outside of your data set. The closer, in magnitude, the correlation coefficient is to 1, the better your model is at explaining your data. Thus, an R^2 value close to 1 tells you that the model is a good model for your data.



Example 1.17 The table below represents the percentage of females (16 or older) that were a part of the civilian labor force in January of the respective years.

Year	1950	1960	1970	1980	1990	2000
Percent of Working Females	33.4	37	43.3	51.6	57.7	60.3

Source: <http://data.bls.gov/servlet/SurveyOutputServlet>

- Find the best-fitting linear model to this data.
- How accurate is the model you found?
- Use your model, to predict the percentage of working females in the year 2010, if this employment trend continues.

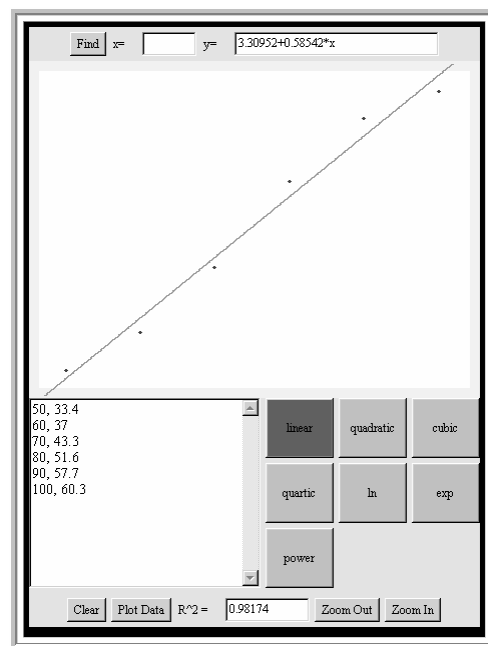


Solution

a. We will let the independent variable, x , represent the year, where the value of x is given as the number of years since 1900. (i.e. 50 represents the year 1950). The dependent variable, y , will represent the percentage of working females. The results of inputting the data into the Modeling Applet and clicking on the linear regression button is shown in Figure 1.19



Figure 1.19 Linear Fit for Example 1.17

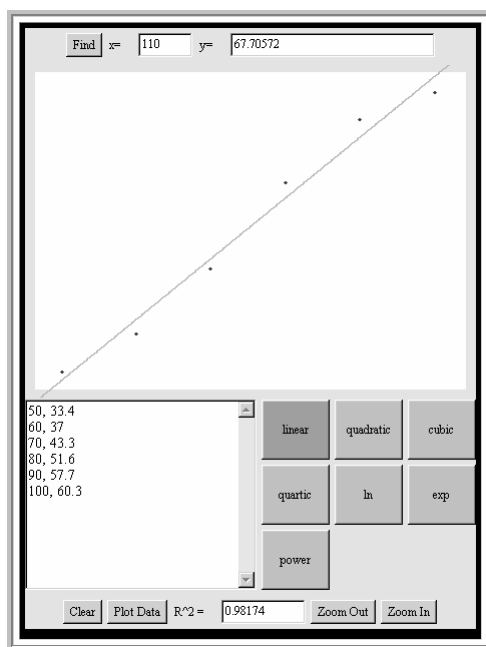


Therefore, $y = 0.58542x + 3.30952$ is the best-fitting linear model to the data.

b. The R^2 value for this model is given as 0.98174, which is very close to 1, and, therefore, shows this model to be an accurate model for the data.

c. What will be happening in the year 2010 is equivalent to finding the y -value corresponding to $x = 110$. Using the Modeling Applet we get


 **Figure 1.20** Modeling Applet with $x = 110$



so, about 67.7% of females will be a part of the civilian labor force, if this trend continues. ◇

When you are not told which model to find (as is the case in real life), you must decide which model best represents the data you are given. Moreover, you are not limited to only linear models. We will restrict our discussion in this section to polynomial models, but you will encounter the use of other models in later chapters, as different functions are introduced.

When trying to determine which model best represents your data, first look at a scatterplot of your data and determine the possible shape your model should have. (Remember to think of the end behavior your model should have for predicting values outside of your given data set.) Second, find all models which fit the shape of your data and plot each model with your data. (Remember to zoom out to see the end behavior of your models.) Last, compare the models to find the best fit. You are looking for the simplest model which accurately predicts the given situation.

 **Example 1.18** The following table gives the 30-year fixed-rate conventional mortgage interest rates from 1972 to 1982.

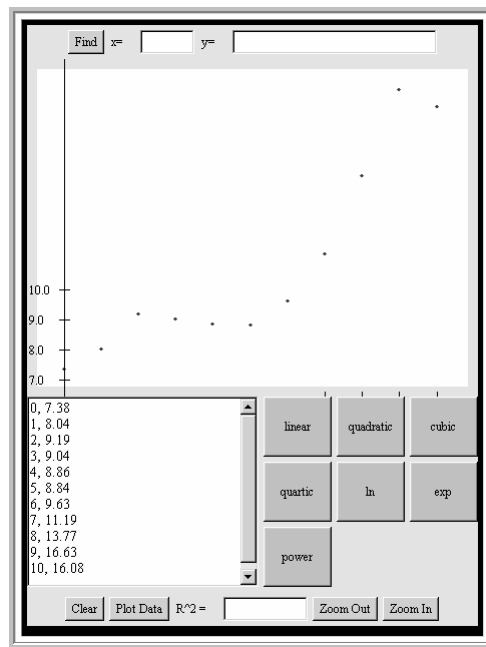
Year	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
Interest Rate	7.38	8.04	9.19	9.04	8.86	8.84	9.63	11.19	13.77	16.63	16.08

Source: <http://www.federalreserve.gov/releases/h15/data/a/cm.txt>

Find the best-fitting polynomial model to this data and explain why the model you chose is the best model.

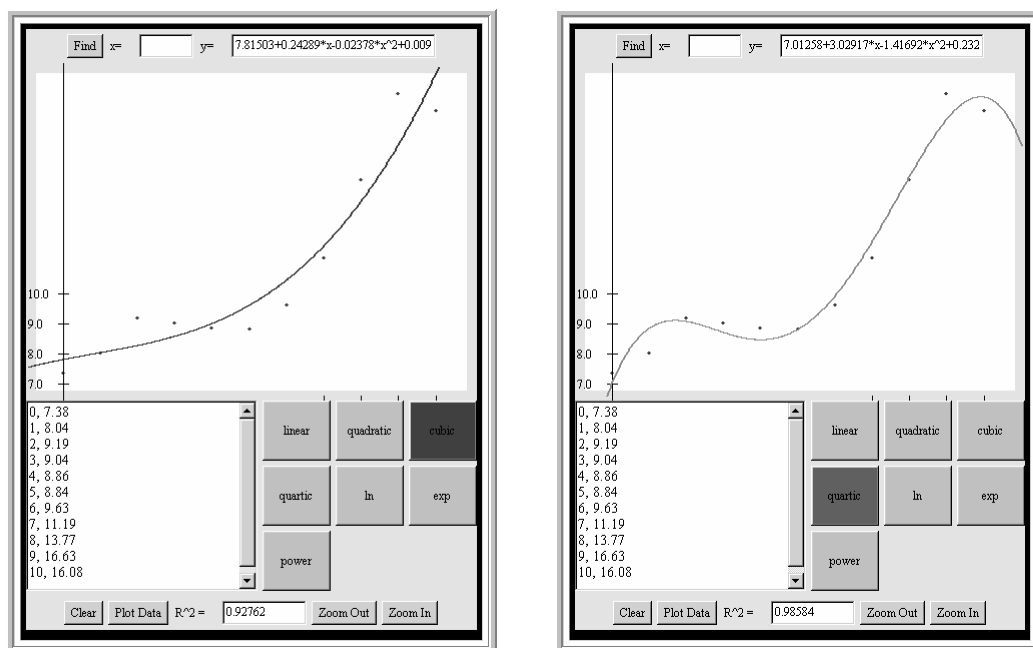
✓ **Solution** We will let the independent variable, x , represent the year, where the value of x is given as the number of years since 1972. (i.e. 0 represents the year 1972). The dependent variable, y , will represent the interest rate. We will first input the data into the Modeling Applet and look at the scatterplot of the data. The results are shown in Figure 1.21.

□ **Figure 1.210** Interest Rate Data



Since the data does not strictly increase, we can rule out using a linear model. Likewise, it begins by increasing and then decreases, but then increases again so a quadratic model would not seem to be the best fit for this data. The only two polynomial models worth trying are cubic and quartic shown below.

 Figure 1.22 Modeling Applet with Cubic and Quadratic Fits.



It is clear that even though quartic is the most complicated polynomial model, it best represents the given data set. Thus, the best-fitting model is

$$y = -0.01114x^4 + 0.23231x^3 - 1.41692x^2 + 3.02917x + 7.01258$$



Sample Quiz

Question 1.1 Given $f(x) = -\frac{2}{3}x + 5$, if x decreases by 9, what is the corresponding change in the function value?

Question 1.2 Small fish bowls sell for \$5.00. The company making the fish bowls has total costs of \$550 when making 50 fish bowls and \$650 when making 100 fish bowls. Find the profit function of the company making the fish bowls.

Question 1.3 At a price of \$40 per book, 500 books can be sold and at a price of \$30 per book, 100 books can be sold.

- Assuming linear demand, find the demand equation as a function of the number of books sold.
- What is the highest price consumers are willing to pay for this book?

Question 1.4 Kathryn bought a brand new car in 1999 for \$20,500. In 2002 it is only worth \$13,285. Assuming linear depreciation is occurring, what will her car be worth in 2006?

Question 1.5 Find the vertex of $f(x) = 3x^2 - 6x + 7$.

Question 1.6 How will the graph of $g(x) = -3(x - 4)^2 + 2$ be different from the graph of $f(x) = x^2$?

Question 1.7 The linear demand function for a particular item is given by $p = -2x + 50$. If the linear cost function for the company producing the item is given by $C(x) = 30x + 40$, find the number of items the company must make and sell in order to maximize its profits.

Question 1.8 Describe the end behavior of the function $f(x) = 6x^{10} - 3x^8 + x^7 - 2x + 1$.

Question 1.9 The following table gives the life expectancy of females at birth (in years) for the given years.

Year	1929	1939	1949	1959	1969	1979	1989	1999
Life Expectancy	58.7	65.4	70.7	73.2	74.4	77.8	78.5	79.4

Source: National Vital Statistics Report, Vol. 50, No. 6, March 21, 2002

Find the equation of the best-fitting quadratic model to this data.

Question 1.10 The following table gives the median income (in dollars) for a four-person family living in Texas between 1990 and 2000.

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Income	37,789	39,204	40,342	40,688	42,570	43,977	46,757	48,007	51,148	53,291	53,513

Source: ????

Which polynomial model (linear, quadratic, cubic, or quartic) best fits the given data and why?