


Chapter 9 Definite Integrals

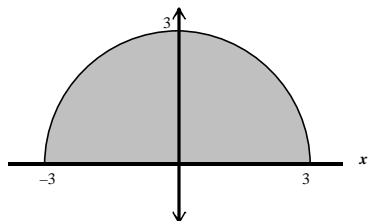
In the previous chapter we found how to take an antiderivative and investigated the indefinite integral. In this chapter the connection between antiderivatives and definite integrals is established as we try to solve one of the most famous problems in mathematics, finding the area under a given curve.

9.1 Approximating Area Under a Curve

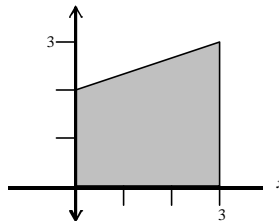
When it comes to finding the area of basic geometric shapes such as circles, squares, rectangles, triangles, and trapezoids, we can rely on geometric formulas to calculate the area.

 **Example 9.1** Find the area of the shaded regions.

a.



b.



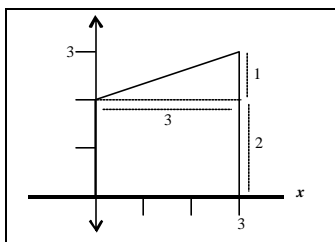
 **Solution**

a. The shaded region is half of a circle with a radius of 3. Thus, we will use the formula for the area of a circle ($A = \pi r^2$), multiply it by 0.5 and use $r = 3$.

$$A_{\text{region}} = .5(\pi(3)^2) = .5(9\pi) = 4.5\pi$$

b. The shaded region is made up of a rectangle ($A = lw, l = 3, w = 2$) and a triangle ($A = \frac{1}{2}bh, b = 3, h = 1$). If we add the areas of the two geometric shapes, we will have found the area of the shaded region.

Figure 9.1 Graph for Example 9.1a

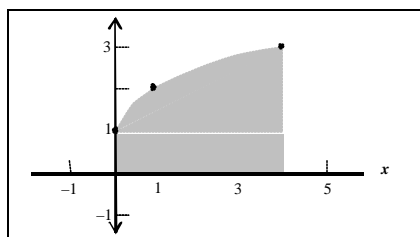


$$A_{\text{shaded}} = A_{\text{rectangle}} + A_{\text{triangle}} = (lw) + \left(\frac{1}{2}bh\right) = (3 \cdot 2) + \left(\frac{1}{2} \cdot 3 \cdot 1\right) = 6 + \frac{3}{2} = \frac{15}{2}$$

The area of the trapezoid is therefore 7.5 square units. ◇

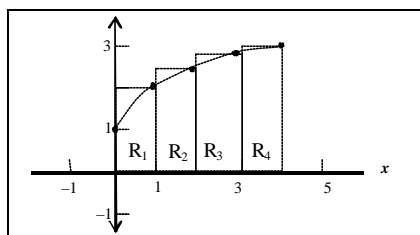
Finding the area between the x -axis and a curve $f(x)$ on a given interval is a bit more challenging if the region formed is not a “basic” geometric shape. For example, the area under the curve $f(x) = \sqrt{x} + 1$ on $[0, 4]$ forms the shape shown in Figure 9.2.

Figure 9.2 Graph of $f(x) = \sqrt{x} + 1$.



We can see from the figure that the area between the x -axis and $f(x)$ is not a shape that has a familiar formula for finding the area. When this occurs, we use rectangles to approximate the area of the region. If we draw four rectangles, as seen in Figure 9.3, we can sum up the area of the rectangles ($R_1 + R_2 + R_3 + R_4$) and obtain an approximation of the area under the curve.

Figure 9.3 Graph of $f(x) = \sqrt{x} + 1$ with 4 right endpoints^H.



^H The rectangles have been constructed such that the right endpoint, x_i , of the interval touches the curve. Because of this, we call these rectangles right endpoint rectangles

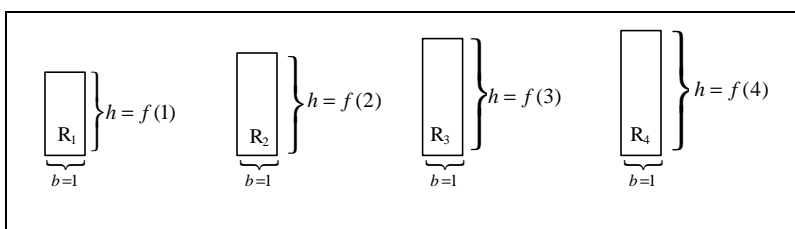
We will, however, find an overestimate of the area because the rectangles extend above the curve. Nonetheless, we will have some idea of the area under the curve.

To find the area of each rectangle in Figure 9.3 we need to find the base and height of each rectangle. The base of each rectangle, Δx , is found by taking the length of the given interval, $x_{\max} - x_{\min}$ and dividing it by the number of rectangles constructed, n . This leads to the following calculation.

$$\Delta x = \frac{x_{\max} - x_{\min}}{n} = \frac{4}{4} = 1.$$

The height of each rectangle is the value of the function from the right end of each interval. Figure 9.4 below shows the dimensions of each rectangle.

Figure 9.4 Dimensions of R_1 , R_2 , R_3 , and R_4 .



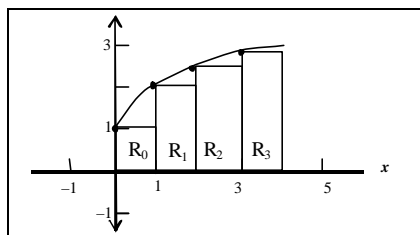
The sum of the rectangles are found in the table below. The area, A , is the base times the height.

Rectangle #	Base, Δx	Right Endpt, x_i	Height, $f(x_i)$	$A = \Delta x \cdot f(x_i)$
R_1	1	1	$f(1) = \sqrt{1} + 1 = 2$	$(1)(2) = 2$
R_2	1	2	$f(2) = \sqrt{2} + 1 \approx 2.41$	$(1)(2.41) = 2.41$
R_3	1	3	$f(3) = \sqrt{3} + 1 \approx 2.73$	$(1)(2.73) = 2.73$
R_4	1	4	$f(4) = \sqrt{4} + 1 = 2 + 1 = 3$	$(1)(3) = 3$
Total Area (Right)				10.14

Since this is an overestimate, the area under the curve is less than 10.14 units.

We can also approximate the area under the curve using left endpoint rectangles as shown in figure 9.4. This approximation will give us an underestimate because the rectangles do not fill the entire area under the curve.

Figure 9.5 Graph of $f(x) = \sqrt{x} + 1$ with 4 left endpoint rectangles. (rename rectangles!!)



The base of each rectangle is still 1 unit but the height of each rectangle is the value of the function from the left end of each interval. The sum of the rectangles is found in the table below.

Rectangle #	Base Δx	Left Endpt, x_i	Height, $f(x_i)$	$A = \Delta x \cdot f(x_i)$
R ₁	1	0	$f(0) = \sqrt{0} + 1 = 1$	$(1)(1) = 1$
R ₂	1	1	$f(1) = \sqrt{1} + 1 = 2$	$(1)(2.41) = 2.41$
R ₃	1	2	$f(2) = \sqrt{2} + 1 \approx 2.41$	$(1)(2.73) = 2.73$
R ₄	1	3	$f(3) = \sqrt{3} + 1 \approx 2.73$	$(1)(3) = 3$
Total Area (Left)				9.14

Thus, the area must be greater than 9.14 units.

In general, we can find an approximation for the area under a continuous curve $f(x)$ on $[a, b]$ by drawing n equally spaced right (or left) endpoint rectangles under the curve and then finding the sum of the area of the rectangles. If Δx is the width of each rectangle and x_i an endpoint where $x_0 = a$ and $x_n = b$, then the sum of the area of n rectangles is

$$A = \left(\begin{array}{c} \text{area of 1}^{\text{st}} \\ \text{rectangle} \end{array} \right) + \left(\begin{array}{c} \text{area of 2}^{\text{nd}} \\ \text{rectangle} \end{array} \right) + \left(\begin{array}{c} \text{area of 3}^{\text{rd}} \\ \text{rectangle} \end{array} \right) + \dots + \left(\begin{array}{c} \text{area of } n^{\text{th}} \\ \text{rectangle} \end{array} \right)$$

For right endpoint rectangles the sum of the area rectangles can be denoted as ^H

$$\text{Total Area (Right)} = \sum_{i=1}^n \Delta x f(x_i) = \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3) + \dots + \Delta x f(x_n)$$

For left endpoint rectangles, the sum of the area of the rectangles can be denoted as

$$\text{Total Area (Left)} = \sum_{i=0}^{n-1} \Delta x f(x_i) = \Delta x f(x_0) + \Delta x f(x_1) + \Delta x f(x_2) + \dots + \Delta x f(x_{n-1})$$

^H The symbol \sum is the notation for “the sum of”

Example 9.2 Approximate the area under the curve $f(x) = x^2 + 2$ on $[0, 2]$ using

a. 4 right endpoint rectangles

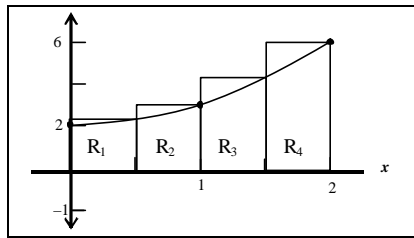
b. 8 left endpoint rectangles.

State if the estimate is an overestimate or an underestimate.

Solution

a. The graph of $f(x) = x^2 + 2$ on $[0, 2]$ with the 4 right endpoint rectangles is shown in Figure 9.6.

Figure 9.6 Graph of 4 right endpoint rectangles



The base of each rectangle is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} = 0.5$$

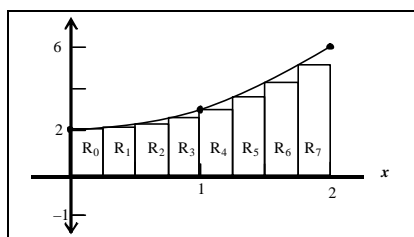
and the height is $f(x_i)$ where x_i is the right endpoint of each interval. The calculation below shows the sum of the areas of the four rectangles.

$$\begin{aligned} \text{Total Area (Right)} &= \left(\sum_{i=1}^4 \Delta x \cdot f(x_i) \right) = 0.5 \cdot f(x_1) + 0.5 \cdot f(x_2) + 0.5 \cdot f(x_3) + 0.5 \cdot f(x_4) \\ &= 0.5 \cdot f(0.5) + 0.5 \cdot f(1) + 0.5 \cdot f(1.5) + 0.5 \cdot f(2) \\ &= 0.5 \left[(0.5)^2 + 2 \right] + 0.5 \left[(1)^2 + 2 \right] + 0.5 \left[(1.5)^2 + 2 \right] + 0.5 \left[(2)^2 + 2 \right] \\ &= 7.75 \end{aligned}$$

Thus, the area under the curve is less than 7.75 square units.

b. The graph of $f(x) = x^2 + 2$ on $[0, 2]$ with the 8 left endpoint rectangles is shown in Figure 9.7.

Figure 9.7 Graph of 8 left endpoint rectangles. (rename rectangles!!)



The base of each rectangle is


$$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = \frac{1}{4} = 0.25$$

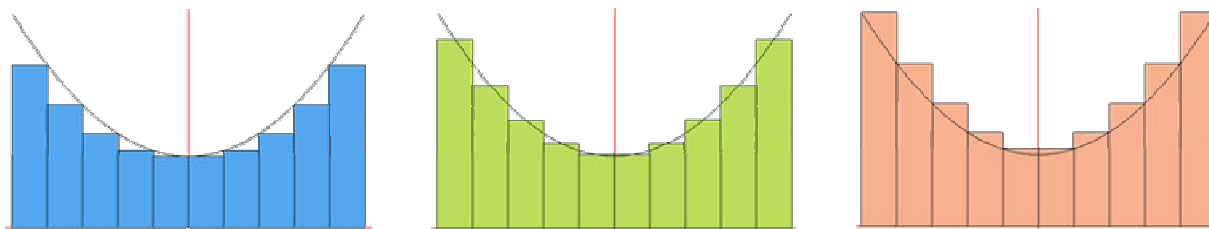
and the height is $f(x_i)$ where x_i is the left endpoint of each interval. The table below shows the sum of the areas of the eight rectangles.

$$\begin{aligned} \text{Total Area (Left)} &= \left(\sum_{i=1}^8 \Delta x \cdot f(x_i) \right) \\ &= 0.5 \cdot f(x_1) + 0.5 \cdot f(x_2) + 0.5 \cdot f(x_3) + 0.5 \cdot f(x_4) + 0.5 \cdot f(x_5) + 0.5 \cdot f(x_6) + 0.5 \cdot f(x_7) + 0.5 \cdot f(x_8) \\ &= 0.5 \cdot f(0) + 0.5 \cdot f(0.25) + 0.5 \cdot f(0.5) + 0.5 \cdot f(0.75) + 0.5 \cdot f(1) + 0.5 \cdot f(1.25) \\ &\quad + 0.5 \cdot f(1.5) + 0.5 \cdot f(1.75) \\ &= 0.5 \left[(0)^2 + 2 \right] + 0.5 \left[(0.25)^2 + 2 \right] + \dots + 0.5 \left[(1.75)^2 + 2 \right] \\ &= 6.1875 \end{aligned}$$

Since these rectangles all lie below the curve, the estimate for the area under the curve is an underestimate. \diamond

There are numerous methods of using rectangles to approximate the area under a curve. A few of the other methods are shown in Figure 9.8 below.

 **Figure 9.8** Other methods to approximate $f(x) = x^2 + 2$ on $[-2, 2]$.



(a) Lower Sum Method – all rectangles lie below the curve.

(b) Midpoint Method – the midpoint of all rectangles are touching the curve.

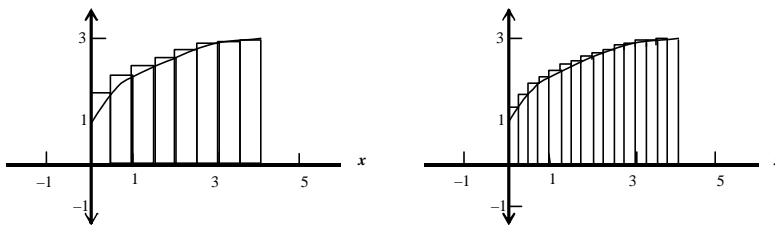
(c) Upper Sum Method – all rectangles lie above the curve.

9.2 Definite Integrals and the Fundamental Theorem of Calculus

The methods used in the previous section allow us to obtain a good approximation of the area under a curve, but can we make this approximation better? If we take thinner and thinner rectangles, we can make the approximation of the area under the curve more accurate. In Example 9.2 we found an approximation of 6.1875 square units for the area under the curve using 8 left endpoint rectangles. If we

would have used 4 left endpoint rectangles our approximation would have been 5.75 square units. The approximation with 8 rectangles was more accurate simply because more rectangles were used. Compare the rectangles in Figure 9.9.

Figure 9.8 Graphs of $f(x) = \sqrt{x} + 1$ with 8 and 16 right endpoint rectangles.



It appears that the amount of excess area made by the 16 rectangles is considerably less than the excess area made by the 8 rectangles. One can imagine that the approximation would be even better if we could fit 100 rectangles or even 1000 rectangles under the curve. What if we had an infinite number of rectangles drawn under the curve? As one might hypothesize, the sum of an infinite number of rectangles does accurately find the area under a curve, and we represent the area under a curve using the definite integral.

The Definite Integral

For a continuous function f on the interval $[a, b]$ ^H let $\Delta x = (b-a)/n$ and x_i be the right endpoint of the n intervals. Then the definite integral of f is

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x \cdot f(x_i)$$

Some useful properties of definite integrals are listed in Table 9.1.

Table 9.1 Properties of Definite Integrals

$$\int_a^b k \cdot f(x) \, dx = k \int_a^b f(x) \, dx$$

$$\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

$$\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx$$

^H Note: “a” is referred to as the lower limit and “b” as the upper limit. Together, “a” and “b” are known as the limits of integration.

Using the definition of the definite integral the area in Figure 9.9 is represented as

$$\int_0^4 (\sqrt{x} + 1) dx$$

 **Example 9.3** Represent the area of the shaded regions from Example 9.1 as definite integrals.

 **Solution**

a. $\int_{-3}^3 \sqrt{9-x^2} dx$

b. $\int_0^3 (\frac{1}{3}x + 2) dx$ ◇

From Example 9.1 (b) we found the area to be exactly $\frac{15}{2}$ and from Example 9.3 (b) we found that the area can be represented as a definite integral. We can put the two of these together and conclude

$$\int_0^3 (\frac{1}{3}x + 2) dx = \frac{15}{2}.$$

Next we can connect the notion of an antiderivative and a definite integral. Take the antiderivative of $\frac{1}{3}x + 2$,

$$F(x) = \frac{1}{3} \left(\frac{1}{2} x^2 \right) + 2x + C = \frac{1}{6} x^2 + 2x + C$$

Note that $F(0) = C$ and $F(3)$ is

$$F(3) = \frac{1}{6}(3^2) + 2(3) + C = \frac{15}{2} + C$$

now find $F(3) - F(0)$,

$$F(3) - F(0) = \frac{15}{2} + C - C = \frac{15}{2}.$$

We can see that finding the antiderivative $F(x)$ of a function and then evaluating $F(b) - F(a)$ gives the exact area under the curve. This process is an important theorem in calculus known as the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

If f is a continuous function defined on a closed interval $[a, b]$ and F is an antiderivative of f , then

$$\int_a^b f(x) dx = F(b) - F(a) = F(x) \Big|_a^b$$

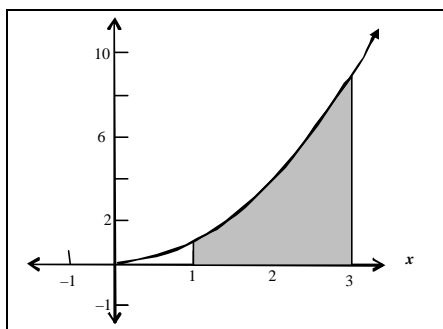
Example 9.4 Draw a geometric representation of each definite integral and then evaluate the definite integral using the Fundamental Theorem of Calculus.

a. $\int_1^3 x^2 dx$

b. $\int_1^e \frac{1}{x} dx$

Solution

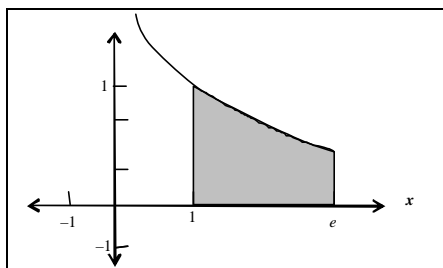
a. The shaded region in the graph below shows the geometric representation.



Use the Fundamental Theorem of Calculus to find the value of the definite integral.

$$\int_1^3 x^2 dx = \left. \frac{1}{3} x^3 \right|_1^3 = \frac{1}{3} (3^3) - \frac{1}{3} (1^3) = 9 - \frac{1}{3} = \frac{26}{3}$$

b. The shaded region in the graph below shows the geometric representation.



Use the Fundamental Theorem of Calculus to find the value of the definite integral.

$$\int_1^e \frac{1}{x} dx = \ln x \Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1$$

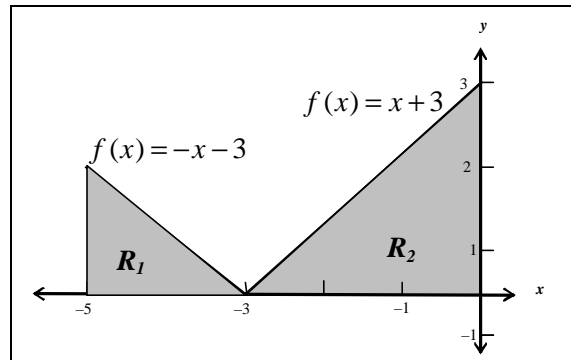


Example 9.5 Graph $f(x) = |x+3|$ and use the graph to find

$$\int_{-5}^0 |x+3| dx .$$



Solution The graph of $f(x) = |x+3|$ is shown below.



To find $\int_{-5}^0 |x+3| dx$, we need to write an integral that represents R_1 and another to represent R_2 . This is necessary because R_1 and R_2 are bounded by different functions.

$$\int_{-5}^0 |x+3| dx = \int_{-5}^{-3} -x-3 dx + \int_{-3}^0 x+3 dx = \left. -\frac{1}{2}x^2 - 3x \right|_{-5}^{-3} + \left. \frac{1}{2}x^2 + 3x \right|_{-3}^0 = 2 + 4.5 = 6.5 \quad \diamond$$

All of the integrals we have considered thus far have been positive. That is the graphs of the functions lied strictly above the x - axis. The next example demonstrates what happens when a shaded region lies strictly below the x - axis.



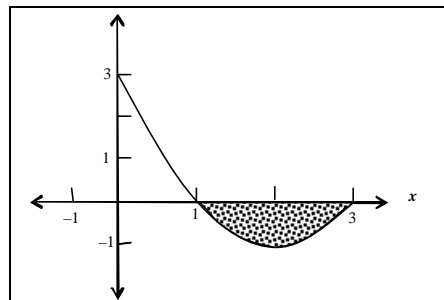
Example 9.6 Draw a geometric representation of

$$\int_1^3 (x^2 - 4x + 3) dx$$

then evaluate the definite integral using the Fundamental Theorem of Calculus.



Solution The shaded region in the graph below shows the geometric representation.



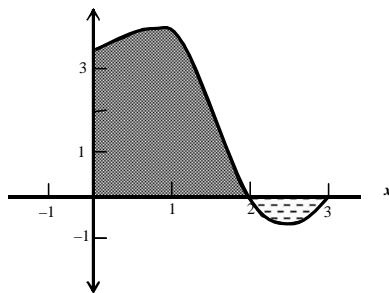
$$\int_1^3 (x^2 - 4x + 3) dx = \left. \frac{1}{3}x^3 - 2x^2 + 3x \right|_1^3 = \left(\frac{1}{3}(3)^3 - 2(3)^2 + 3(3) \right) - \left(\frac{1}{3}(1)^3 - 2(1)^2 + 3(1) \right)$$

$$= (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3 \right) = -\frac{4}{3}$$

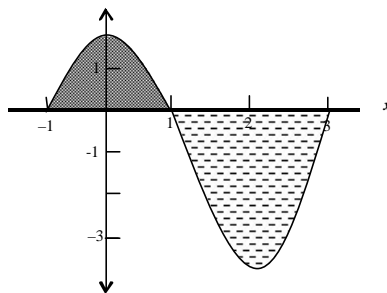
This definite integral is negative because the shaded area lies below the x -axis. ◇

When a definite integral represents a portion of the graph that lies above as well as below the x -axis we can calculate two types of areas, **gross area** and **net area**. The gross area is the total amount of area that lies between the curve and the x -axis while the net area calculates how much more area lies above or below x -axis. Figure 9.7 shows the different values of the net area.

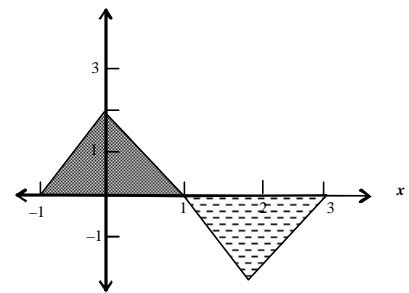
Figure 9.?? Net Area.



Net area is positive because more area lies above the x -axis.



Net area is negative because more area lies below the x -axis



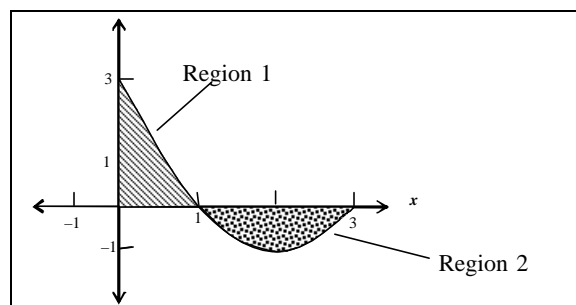
Net area is zero because the area above and below the x -axis is the same.

Example 9.7 Draw a geometric representation of

$$\int_0^3 (x^2 - 4x + 3) dx$$

and then calculate the net and gross areas.

Solution The shaded region in the graph below shows the geometric representation.



To find the gross area we need to evaluate the integral that represents each shaded region.

$$\text{Region}_1 = \int_0^1 (x^2 - 4x + 3) dx = \left. \frac{1}{3}x^3 - 2x^2 + 3x \right|_0^1 = \left(\frac{1}{3} - 2 + 3 \right) - 0 = \frac{4}{3}$$

In Example 9.6, we found the area of region 2 to be $-\frac{4}{3}$. Therefore the gross area is

$$\begin{aligned} \text{Area of Region}_1 + |\text{Area of Region}_2| &= \int_0^1 (x^2 - 4x + 3) dx + \left| \int_1^3 (x^2 - 4x + 3) dx \right| \\ &= \frac{4}{3} + \left| -\frac{4}{3} \right| = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \end{aligned}$$

The net area is just the sum of the two integrals,

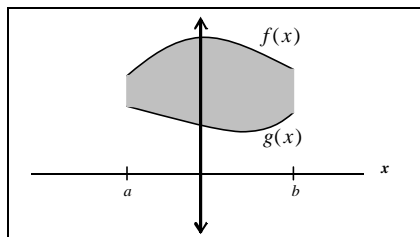
$$\int_0^3 (x^2 - 4x + 3) dx = \int_0^1 (x^2 - 4x + 3) dx + \int_1^3 (x^2 - 4x + 3) dx = \frac{4}{3} + \left(-\frac{4}{3} \right) = 0.$$

Since the net area is zero, we know there is the same amount of area above the x -axis as there is below the x -axis. Notice that calculating the function over the entire interval is another method of obtaining net area.

9.3 Area Between Two Curves

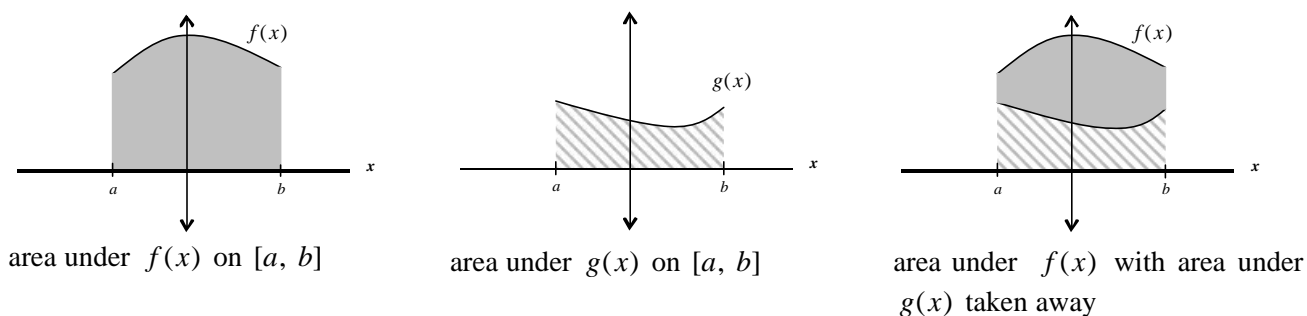
Suppose we are to find the area of the shaded region shown in Figure 9.??.

Figure 9.?? The area between $f(x)$ and $g(x)$.



The area under $f(x)$ on $[a, b]$ is shown in Figure 9.?? (a) and the area under $g(x)$ on $[a, b]$ is shown in Figure 9.?? (b). If the area under $g(x)$ is taken away from the area under $f(x)$ we obtain the area in Figure 9.?? (c) which is the area we were trying to find in Figure 9.??.

Figure 9.?? Area Between Two Curves



Thus, we can find the area between two curves if we find the area under the top curve and subtract off the area under the bottom curve.

Area Between Two Curves

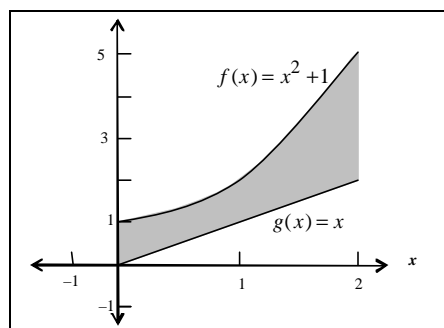
On the closed interval $[a, b]$, the area between two continuous functions $f(x)$ and $g(x)$, where $f(x) \geq g(x)$, is given by

$$\int_a^b [f(x) - g(x)] dx$$

The area between two curves can be remembered as $\int_a^b (\text{top function} - \text{bottom function}) dx$

Example 9.8 Find the area between $f(x) = x^2 + 1$ and $g(x) = x$ on $[0, 2]$.

Solution First lets graph both functions over $[0, 2]$.

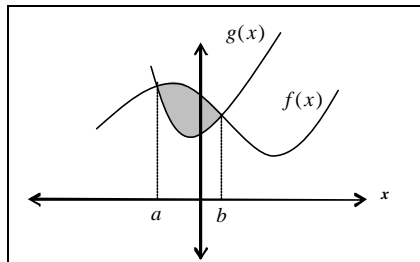


Since $f(x)$ is the top function and $g(x)$ is the bottom function, the definite integral, and thus the area between the two curves is

$$\int_0^2 [(x^2 + 1) - x] dx = \int_0^2 (x^2 - x + 1) dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + x \right]_0^2 = \frac{1}{3}(2)^3 - \frac{1}{2}(2)^2 + 2 - 0 = \frac{8}{3} - 2 + 2 = \frac{8}{3}$$

Sometimes the two given curves will intersect at one or more points, thus forming an area bounded by the curves as shown in Figure 9.??.

Figure 9.??

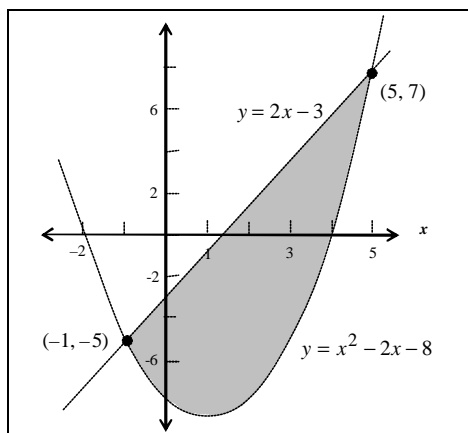


To find the area bounded by two curves we need to find the limits of integration. We do this by locating the points where the curves intersect. The definite integral for Figure 9.10 is represented by

$$\int_a^b [f(x) - g(x)] dx$$

Example 9.9 Find the area of the region bounded by $y = x^2 - 2x - 8$ and $y = 2x - 8$.

Solution First, we need to graph the two functions on the same coordinate plane.

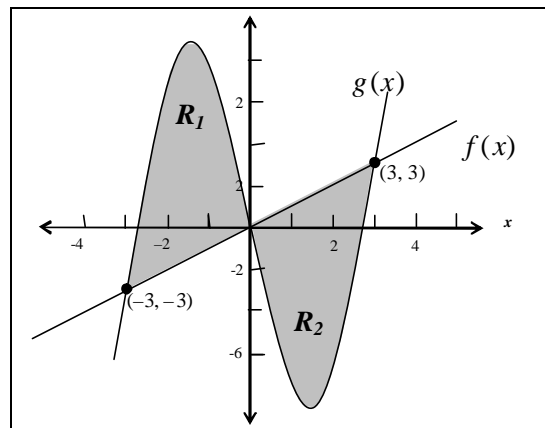


From the graph we notice that $y = 2x - 3$ is the top function and $y = x^2 - 2x - 8$ is the bottom function. In addition, the points of intersection show that the lower limit of integration is $x = -1$ and the upper limit of integration is $x = 5$. Thus, the definite integral is

$$\begin{aligned}
\int_{-1}^5 (2x-3) - (x^2 - 2x - 8) \, dx &= \int_{-1}^5 2x - 3 - x^2 + 2x + 8 \, dx = \int_{-1}^5 -x^2 + 4x + 5 \, dx \\
&= \left[-\frac{1}{3}x^3 + 2x^2 + 5x \right]_{-1}^5 = -\frac{1}{3}(125) + 2(25) + 5(5) - \left[-\frac{1}{3}(-1) + 2(1) + 5(-1) \right] \\
&= \frac{100}{3} + \frac{8}{3} = \frac{108}{3} = 36
\end{aligned}$$

Example 9.10 Find the area of the region bounded by $f(x) = x$ and $g(x) = x^3 - 8x$.

Solution First, we need to graph the two functions on the same coordinate plane.



There are two bounded regions (R_1 and R_2) produced by these curves. Notice that the top function of R_1 is $g(x)$ and the top function of R_2 is $f(x)$. Consequently we will need to set up an integral to find the area of R_1 , another integral to find the area of R_2 , and then add the results.

$$\begin{aligned}
R_1 &= \int_{-3}^0 (x^3 - 8x - x) \, dx \\
&= \int_{-3}^0 (x^3 - 9x) \, dx \\
&= \left[\frac{1}{4}x^4 - \frac{9}{2}x^2 \right]_{-3}^0 \\
&= 0 - \left(\frac{1}{4}(81) - \frac{9}{2}(9) \right) \\
&= \frac{81}{4} = 20.25
\end{aligned}$$

$$\begin{aligned}
R_2 &= \int_0^3 (x^3 - 8x - x) \, dx \\
&= \int_0^3 (x^3 - 9x) \, dx \\
&= \left[\frac{1}{4}x^4 - \frac{9}{2}x^2 \right]_0^3 \\
&= \frac{1}{4}(81) - \frac{9}{2}(9) \\
&= \frac{81}{4} = 20.25
\end{aligned}$$

Now adding the two results together we get

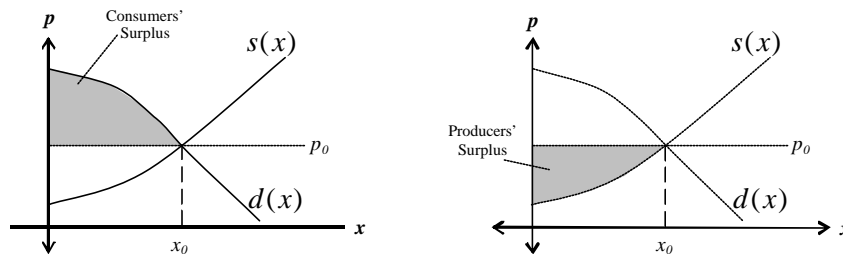
$$R_1 + R_2 = 20.25 + 20.25 = 40.5$$

9.4 Applications of Definite Integrals

Consumers' and Producers' Surplus

Suppose you worked all summer and put away \$800 to buy a new stereo system for your dorm room. When you went shopping to buy the stereo system you found exactly what you wanted for only \$650. Thus, we could say that you “saved” \$150. If we could find all the consumers who were willing to pay over \$650 for this stereo system and calculate the total savings of all consumers, we will have found the **consumers' surplus**. Figure 9.11(a) shows the graph of a supply curve, $p = s(x)$, and a demand curve, $p = d(x)$. The dotted lines represents the equilibrium price, p_0 , and the equilibrium quantity x_0 . The area above the dotted line, but below $d(x)$, would represent the consumers' surplus.

Figure 9.??



Now lets say you are the producer of the stereo systems and are willing to supply the stereos for \$500. If, however, you end up selling the stereos for \$650, you have “gained” \$150. The total amount gained over all possible prices is the **producers' surplus**. Figure 9.11 also shows the graph of the producers' surplus.

If $p = d(x)$ is the demand equation, $p = s(x)$ the supply equation, and (x_0, p_0) is the equilibrium point then the consumers' surplus is given by

$$\int_0^{x_0} (d(x) - p_0) dx$$

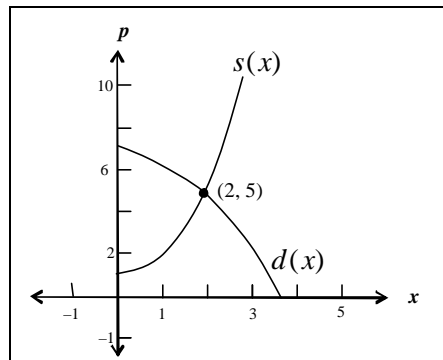
the producers' surplus is given by

$$\int_0^{x_0} (p_0 - s(x)) dx$$



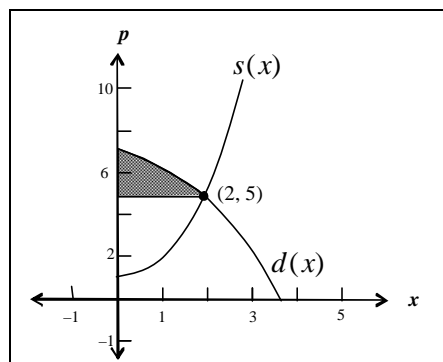
Example 9.11 A company has determined that its supply and demand equations can be modeled by $p = d(x) = -\frac{1}{2}x^2 + 7$ and $p = s(x) = x^2 + 1$ where x represents the number of units supplied each week and p is the selling price (in hundreds of dollars) for each unit. Find the consumers' and producers' surplus.

✓ **Solution** First we need to graph the supply and demand functions and find the equilibrium point. The equilibrium point is found by setting $d(x) = p(x)$.



$$\begin{aligned}
 d(x) &= s(x) \\
 -\frac{1}{2}x^2 + 7 &= x^2 + 1 \\
 6 &= \frac{3}{2}x^2 \\
 12 &= 3x^2 \\
 4 &= x^2 \\
 2 &= x
 \end{aligned}$$

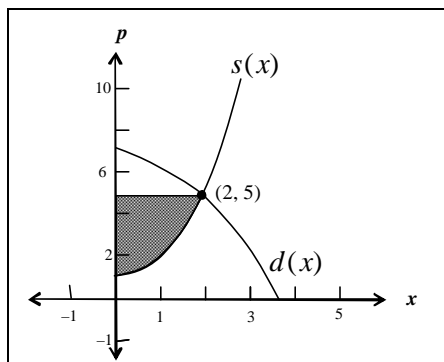
The consumers' surplus is



$$\int_0^2 \left(-\frac{1}{2}x^2 + 7 - 5 \right) dx = \int_0^2 \left(-\frac{1}{2}x^2 + 2 \right) dx = \left[-\frac{1}{6}x^3 + 2x \right]_0^2 = -\frac{1}{6}(8) + 2(2) = -\frac{4}{3} + 4 = \frac{8}{3}$$

So the consumers' "saved" approximately \$266.67 per week when the selling price was \$500.

The producers' surplus is



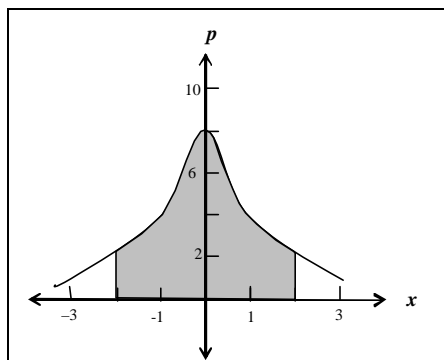
$$\int_0^2 (5 - (x^2 + 1)) dx = \int_0^2 (-x^2 + 4) dx = \left[-\frac{1}{3}x^3 + 4x \right]_0^2 = -\frac{1}{3}(8) + 4(2) = -\frac{8}{3} + 8 = \frac{16}{3}$$

So the producers' "saved" approximately \$533.33 per week when the selling price was \$500. ◇

Sample Quiz

Question 9.1 Find an approximation for the area under $f(x) = -2x^2 + 2$ on $[-1, 0]$ using 4 left endpoint rectangles and 4 right endpoint rectangles. Which is an overestimate and which is an underestimate?

Question 9.2 Write a definite integral that represents the shaded area.



Question 9.3 Evaluate $\int_{-2}^2 -x^2 + 4 dx$.

Question 9.4 Draw a graph of $f(x) = |x - 2|$ and then find $\int_{-1}^4 |x - 2| dx$.

Question 9.5 Evaluate $\int_{-1}^2 x^3 - x^2 - 4 dx$.

Question 9.6 Calculate the net and gross areas of $\int_1^8 x^2 - 10x + 21 \, dx$.

Question 9.7 Find the area between $f(x) = \frac{1}{2}x + 1$ and $g(x) = x^2 - 4x + 9$ on $[3, 6]$.

Question 9.8 Find area bounded by $f(x) = x^2 + 4x + 2$ and $g(x) = 4x + 6$

Question 9.9 Find the area bounded by $f(x) = x^3 - 9x$ and $g(x) = \frac{1}{2}x$.

Question 9.10 A company has determined its demand equation can be modeled by $p = d(x) = -0.25x^2 + 80$ and its supply equation can be modeled by $p = s(x) = 3.5x + 20$ where x is the number of units sold per day and p is the selling price in hundreds of dollars. Find the consumers' and producers' surplus.