Direct Serendipity and Mixed Finite Elements on Quadrilaterals
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Background and Objective

Serendipity finite elements. \( S_r \), on rectangle \( E \):
- \( H^1 \)-conforming
- Approximate to \( O(h^{r+1}) \) with minimal # of degrees of freedom (DoFs)

BDM mixed finite elements. \( \text{BDM}_r \), on rectangle \( E \):
- \( H(\text{div}) \)-conforming
- Approximate velocity to \( O(h^{r+1}) \) with minimal # of DoFs

Related by de Rham complex.

\[ \mathbb{R} \hookrightarrow S_{r+1}(E) \xrightarrow{\text{curl}} \text{BDM}_r(E) \xrightarrow{\text{div}} P_{r-1}(E) \rightarrow 0 \]

Problem. Lose accuracy when mapped to a quadrilateral \( E \)

Objective. Define direct finite element spaces that
- Include \( P_r(E) \) directly in the space (for approximation)
- Use minimal number of degrees of freedom

Minimal # DoFs for \( H^1 \)-Conformity \((r \geq 2)\)

<table>
<thead>
<tr>
<th>Geometric Object</th>
<th>Number</th>
<th>DoFs/Object</th>
<th>Total DoFs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 vertex</td>
<td>4</td>
<td>dim ( P_r )</td>
<td>( 4(r-1) )</td>
</tr>
<tr>
<td>1 edge</td>
<td>4</td>
<td>dim ( P_{r-1} )</td>
<td>( \frac{1}{2}(r-2)(r-3) )</td>
</tr>
<tr>
<td>total # DoFs</td>
<td>dim ( P_r ) + 2 ( \Rightarrow ) We must add 2 supplements to ( P_r )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Direct Serendipity Spaces

\[ \text{DS}_r(E) = P_r(E) \oplus G_{\text{DS}}^r(E) \]

We get a family of direct serendipity elements for supplements

\[ G_{\text{DS}}^r(E) = \text{span}\{ \lambda_2 \lambda_3 \lambda_4^{r-2} \lambda_1 R_{13}, \lambda_1 \lambda_3 \lambda_5^{r-2} \lambda_2 R_{24} \} \]

Choices.
1. Linear functions \( \lambda_{24} \) and \( \lambda_{13} \)

\[ \lambda_{24}(x) = -(x - x_{24}^*) \cdot v_{24} \quad \text{and} \quad \lambda_{13}(x) = -(x - x_{13}^*) \cdot v_{13} \]

2. The functions \( R_{13} \) and \( R_{24} \) are defined to satisfy the properties

\[ R_{13}(x)|_{x_1} = -1, \quad R_{13}(x)|_{x_3} = 1, \quad R_{24}(x)|_{x_2} = -1, \quad R_{24}(x)|_{x_4} = 1 \]

Further define: \( R_1 = \frac{1}{2}(1 - R_{13}), \quad R_2 = \frac{1}{2}(1 + R_{13}) \), etc.

Nodal Basis Functions

Interior nodal basis functions \((r \geq 4)\)

Let \( \{ \varphi_{E,i} \} \subset \text{P}_{r-4} \) be a nodal basis for the cell nodes \( \{ x_{E,i} \} \), where \( i = 1, \ldots, \dim \text{P}_{r-4} \)

\[ \varphi_{E,i}(x) = \frac{[\lambda_1 \lambda_2 \lambda_3 \lambda_4](x) \delta_{E,i}(x)}{[\lambda_1 \lambda_2 \lambda_3 \lambda_4](x_{E,i})}, \quad i = 1, \ldots, \dim \text{P}_{r-4} \]

Edge nodal basis functions, e.g., \( \varphi_{E,11}(x) \) is 1 at \( x_{E,11} \), 0 at other nodes
For some \( p \in \text{P}_{r-3}(E) \), (take \( p = 0 \) if \( r = 2 \)), let

\[ \phi_{E,11}(x) = \lambda_2 \lambda_3 \lambda_4 \lambda_5 x_{E,11}^* \cdot v_{E,11} \in \text{DS}_r(E), \]

with \( p \) satisfying the conditions

\[ p(x_{E,11}) = \frac{\lambda_2 \lambda_3 \lambda_4 \lambda_5 x_{E,11}^* \cdot v_{E,11}}{\lambda_1 \lambda_3 \lambda_5}(x_{E,11}) \quad \forall i = 2, \ldots, r - 1 \]

Subtract interior DoFs and normalize.

Vertex nodal basis functions. For example,

\[ \phi_1(x) = \lambda_1^3(x) \cdot v_1 \]

Subtract previous DoFs and normalize.

Approximation Properties of \( \text{DS}_r \)

We define a nodal interpolation operator \( I^n_r \) (cf. Scott & Zhang, 1990)

Theorem.
Assume
- \( 1 \leq p \leq \infty \) and \( l > 1/p \) (or \( 1 \geq p = 1 \))
- \( T_{E,i} \) is uniformly shape regular with parameter \( s \)
- For every \( E \in \mathcal{T}_h \), the zero set of \( \lambda_{24} \) intersects \( e_1 \) and \( e_2 \), and that of \( \lambda_{13} \) intersects \( e_3 \) and \( e_4 \)
- \( R_{13} \) and \( R_{24} \) are \( m \) times differentiable in the vertices of \( E \)

Then there exists \( C = C(r, s) > 0 \), such that \( \forall v \in W_{h}^p(\Omega) \)

\[ \left( \sum_{E \in \mathcal{T}_h} \| v - I^n_r v \|^p \| w_{h}^p(E) \| \right)^{1/p} \leq C h^{l-m} \| v \|^p \| w_{h}^p(\Omega) \| \quad 0 \leq m \leq l \leq r+1 \]

Summary and Conclusions
1. Many families of direct serendipity spaces found for quadrilaterals
   - No mappings required
   - Constructed an explicit basis
   - Optimal order of approximation and number of DoFs
2. New direct mixed finite element spaces found for quadrilaterals
   - Arise from the de Rham theory
   - Optimal order of approximation and number of DoFs
   - We found the direct serendipity space giving AC spaces
3. The construction could be extended to polygonal meshes.

Direct Mixed Spaces

We get mixed finite elements from direct serendipity by de Rham.

\[ \text{Reduced } H(\text{div}) \text{-approximation} \]

\[ \mathbb{R} \hookrightarrow \text{DS}_{r+1}(E) \xrightarrow{\text{curl}} \text{V}^\text{red}_{r}(E) \xrightarrow{\text{div}} P_{r-1}(E) \rightarrow 0 \]

The image of one map is the kernel of the next. DoFs map properly.

\[ \text{V}^\text{red}_{r}(E) = \text{curl } \text{DS}_{r+1}(E) \oplus \text{R}_{r-2} \]

where we identify \( \text{V}^\text{red}_{r}(E) = \text{curl } \text{DS}_{r+1}(E) \)

Full \( H(\text{div}) \)-approximation

\[ \mathbb{R} \hookrightarrow \text{DS}_{r+1}(E) \xrightarrow{\text{curl}} \text{V}^\text{full}_{r}(E) \xrightarrow{\text{div}} P_{r-1}(E) \rightarrow 0 \]

\[ \text{V}^\text{full}_{r}(E) = \text{curl } \text{DS}_{r+1}(E) \oplus \text{R}_{r-2} = P_{r-1}^2(E) \oplus \text{S}^r_{r}(E) \]

AC spaces. We recover the AC spaces [Arbogast & Correa 2016] by a special choice of supplements \( \text{S}^r_{r}(E) \) that are mapped.

Numerical Results

Test problem. \( \Omega = [0, 1]^2 \)

\[ -\nabla \cdot (\nabla p) = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial \Omega \]

Mesh.

Results. \( L^2 \)-errors and convergence rates for \( S_r \) and \( \text{DS}_r \) spaces

<table>
<thead>
<tr>
<th>( n )</th>
<th>( r = 2 )</th>
<th>( r = 3 )</th>
<th>( r = 4 )</th>
<th>( r = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>5.714±0.04</td>
<td>2.94</td>
<td>2.89</td>
<td>3.74</td>
</tr>
<tr>
<td>32</td>
<td>9.799±0.06</td>
<td>2.96</td>
<td>2.85</td>
<td>3.16</td>
</tr>
<tr>
<td>64</td>
<td>1.440±0.06</td>
<td>2.70</td>
<td>2.61</td>
<td>3.05</td>
</tr>
</tbody>
</table>

\( \text{Dof}\)

\[ 8 \quad 3.492±0.04 \quad 3.00 \quad 4.09 \quad 5.00 \quad 8.966±0.8 \quad 5.96 \]

\[ 12 \quad 1.036±0.04 \quad 3.00 \quad 4.08 \quad 5.00 \quad 7.870±0.09 \quad 5.98 \]

\[ 24 \quad 1.296±0.05 \quad 3.00 \quad 4.05 \quad 5.00 \quad 1.335±0.10 \quad 6.08 \]

Similarly, \( \| p - p_h \|_1 \leq C h \) for \( \text{DS}_r \).

Numerical tests for direct mixed spaces show the error and convergence rate for \( \| p - p_h \|, \| u - u_h \|, \) and \( \| \nabla \cdot (u - u_h) \| \) are optimal.
