Direct Serendipity and Mixed Finite Elements on Quadrilaterals

Todd Arbogast, Zhen Tao, and Chuning Wang

Center for Subsurface Modeling, Oden Institute for Computational Engineering & Sciences, University of Texas at Austin

Supported by the NSF

Background and Objective

Serendipity finite elements. $\mathcal{S}_r$ on rectangle $\hat{E}$:

- H^1-conforming
- Approximate to $\mathcal{O}(h^{r+1})$ with minimal # degrees of freedom (DoFs)

BDM mixed finite elements. BDM_r on rectangle $\hat{E}$:

- $H(\mathrm{div})$-conforming
- Approximate velocity to $\mathcal{O}(h^{r+1})$ with minimal # of DoFs

Related by de Rham complex.

$$\mathbb{R} \hookrightarrow \mathcal{S}_{r+1}(\hat{E}) \xrightarrow{\text{curl}} \mathrm{BDM}_r(\hat{E}) \xrightarrow{\text{div}} \mathbb{P}_{r-1}(\hat{E}) \longrightarrow 0$$

Problem. Lose accuracy when mapped to a quadrilateral E

Objective. Define direct finite element spaces that

- Include $\mathbb{P}_r(E)$ *directly* in the space (for approximation)
- Use minimal number of degrees of freedom

Minimal # DoFs for H^1-Conformity ($r \geq 2$)

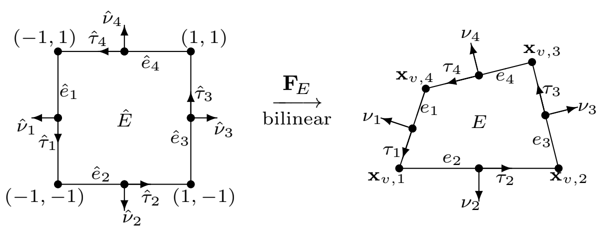
Geometric decomposition.

Dimension	Object	Number	DoFs/Object	Total DoFs
0	vertex	4	1	4
1	edge	4	$\dim \mathbb{P}_{r-2}(e)$	$4(r-1)$
2	cell	1	$\dim \mathbb{P}_{r-4}(E)$	$\frac{1}{2}(r-2)(r-3)$

total # DoFs $= \dim \mathbb{P}_r + 2 \implies$ We must add 2 supplements to $\mathbb{P}_r$

Notation

Bilinear map.



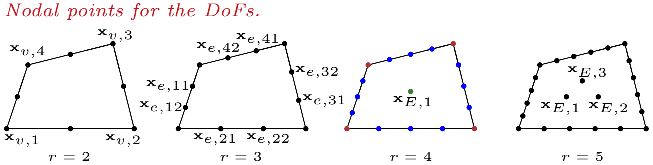
Linear polynomials.

$$\lambda_i(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_i) \cdot \nu_i \quad \propto \quad \text{distance of } \mathbf{x} \text{ to the line through edge } e_i$$

$$\implies \lambda_i\big|_{e_i} = 0$$

Space of polynomials. $\mathbb{P}_r(\Omega)$, $\quad \Omega \subset \mathbb{R}^d$

Nodal points for the DoFs.



Direct Serendipity Spaces

$$\mathcal{DS}_r(E) = \mathbb{P}_r(E) \oplus \mathbb{S}_r^{\mathcal{DS}}(E)$$

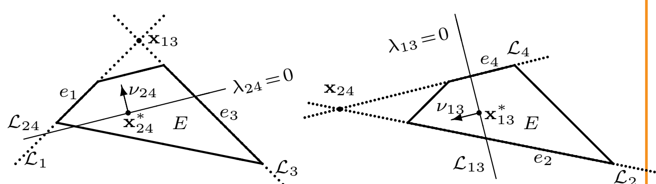
We get a family of direct serendipity elements for supplements

$$\mathbb{S}_r^{\mathcal{DS}}(E) = \mathrm{span}\{\lambda_2\lambda_4\lambda_{24}^{r-2}R_{13}, \lambda_1\lambda_3\lambda_{13}^{r-2}R_{24}\}$$

Choices.

1. Linear functions λ_{24} and λ_{13}

$$\lambda_{24}(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_{24}^*) \cdot \nu_{24} \quad \text{and} \quad \lambda_{13}(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_{13}^*) \cdot \nu_{13}$$



2. The functions R_{13} and R_{24} are defined to satisfy the properties

$$R_{13}(\mathbf{x})|_{e_1} = -1, \quad R_{13}(\mathbf{x})|_{e_3} = 1, \quad R_{24}(\mathbf{x})|_{e_2} = -1, \quad R_{24}(\mathbf{x})|_{e_4} = 1$$

Further define: $R_1 = \frac{1}{2}(1 - R_{13}), R_3 = \frac{1}{2}(1 + R_{13})$, etc.

Nodal Basis Functions

Interior nodal basis functions ($r \geq 4$) Let $\{\phi_{E,i}\} \subset \mathbb{P}_{r-4}$ be a nodal basis for the cell nodes $\{\mathbf{x}_{E,i}\}$, where $i = 1, \dots, \dim \mathbb{P}_{r-4}$

$$\varphi_{E,i}(\mathbf{x}) = \frac{[\lambda_1\lambda_2\lambda_3\lambda_4](\mathbf{x})\phi_{E,i}(\mathbf{x})}{[\lambda_1\lambda_2\lambda_3\lambda_4](\mathbf{x}_{E,i})}, \quad i = 1, \dots, \dim \mathbb{P}_{r-4}$$

Edge nodal basis functions, e.g., $\varphi_{e,11}(\mathbf{x})$ is 1 at $\mathbf{x}_{e,11}$, 0 at other nodes

For some $p \in \mathbb{P}_{r-3}(E)$ (take $p = 0$ if $r = 2$), let

$$\phi_{e,11} = \lambda_2\lambda_4\big(\lambda_3 p + \lambda_{24}^{r-2}R_1\big) \in \mathcal{DS}_r(E),$$

with p satisfying the conditions

$$p(\mathbf{x}_{e,1i}) = -\frac{\lambda_{24}^{r-2}(\mathbf{x}_{e,1i})}{\lambda_3(\mathbf{x}_{e,1i})}, \quad \forall i = 2, \dots, r-1$$

Subtract interior DOFs and normalize.

Vertex nodal basis functions. For example,

$$\phi_{v,1}(\mathbf{x}) = \lambda_3(\mathbf{x})\,\lambda_4(\mathbf{x})$$

Subtract previous DOFs and normalize.

Approximation Properties of $\mathcal{DS}_r$

We define a nodal interpolation operator $\mathcal{I}_h^r$ (cf. Scott & Zhang, 1990)

Theorem. Assume that

- $1 \leq p \leq \infty$ and $l > 1/p$ (or $l \geq 1$ if $p = 1$)
- $\mathcal{T}_h$ is uniformly shape regular with parameter σ_*
- For every $E \in \mathcal{T}_h$, the zero set of λ_{24} intersects e_1 and e_3, and that of λ_{13} intersects e_2 and e_4
- R_{13} and R_{24} are m times differentiable in the vertices of E

Then there exists $C = C(r, \sigma_*) > 0$, such that $\forall v \in W_p^l(\Omega)$,

$$\Big(\sum_{E \in \mathcal{T}_h} ||v - \mathcal{I}_h^r v||_{W_p^m(E)}^p\Big)^{1/p} \leq C\, h^{l-m}\, |v|_{W_p^l(\Omega)}, \quad 0 \leq m \leq l \leq r+1$$

Direct Mixed Spaces

We get mixed finite elements from direct serendipity by de Rham.

Reduced $H(div)$-approximation

$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+1}(E) \xrightarrow{\text{curl}} \mathbf{V}_r^{\text{red}}(E) \xrightarrow{\text{div}} \mathbb{P}_{r-1}(E) \longrightarrow 0$$

The image of one map is the kernel of the next. DoFs map properly.

$$\mathbf{V}_r^{\text{red}}(E) = \text{curl}\,\mathcal{DS}_{r+1}(E) \oplus \mathbf{x}\mathbb{P}_{r-1} = \mathbb{P}_r^2(E) \oplus \mathbb{S}_r^{\mathbf{V}}(E)$$

where we identify $\mathbb{S}_r^{\mathbf{V}}(E) = \text{curl}\,\mathbb{S}_{r+1}^{\mathcal{DS}}(E)$

Full $H(div)$-approximation

$$\mathbb{R} \hookrightarrow \mathcal{DS}_{r+1}(E) \xrightarrow{\text{curl}} \mathbf{V}_r^{\text{full}}(E) \xrightarrow{\text{div}} \mathbb{P}_r(E) \longrightarrow 0$$

$$\mathbf{V}_r^{\text{full}}(E) = \text{curl}\,\mathcal{DS}_{r+1}(E) \oplus \mathbf{x}\mathbb{P}_r = \mathbb{P}_r^2(E) \oplus \mathbf{x}\tilde{\mathbb{P}}_r \oplus \mathbb{S}_r^{\mathbf{V}}(E)$$

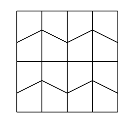
AC spaces. We recover the AC spaces (Arbogast & Correa 2016) by a special choice of supplements $\mathbb{S}_r^{\mathcal{DS}}(E)$ that are mapped.

Numerical Results

Test problem. $\Omega = [0,1]^2$

$$-\nabla \cdot (\nabla p) = f \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega$$

Mesh.



$\mathcal{T}_h$

Results. L^2-errors and convergence rates for $\mathcal{S}_r$ and $\mathcal{DS}_r$ spaces

	$r=2$		$r=3$	$r=4$	$r=5$	
n	error	rate	rate	rate	error	rate
			$\mathcal{S}_r$			
8	5.714e-04	2.92	2.89	3.72	2.005e-06	4.13
32	9.799e-06	2.90	2.85	3.16	8.342e-09	3.84
64	1.440e-06	2.70	2.61	3.05	6.644e-10	3.56
			$\mathcal{DS}_r$			
8	3.492e-04	3.00	4.07	5.00	8.896e-08	5.96
12	1.036e-04	3.00	4.08	4.99	7.870e-09	5.98
24	1.296e-05	3.00	4.05	5.00	1.235e-10	6.00

Similarly, $|p - p_h|_{H^1} \sim \mathcal{O}(h^r)$ for $\mathcal{DS}_r$.

Numerical tests for direct mixed spaces show the error and convergence rate for $||p - p_h||$, $\|\mathbf{u} - \mathbf{u}_h\|$, and $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|$ are optimal.

Summary and Conclusions

1. Many families of direct serendipity spaces found for quadrilaterals
   - No mappings required
   - Constructed an explicit basis
   - Optimal order of approximation and number of DOFs
2. New direct mixed finite element spaces found for quadrilaterals
   - Arise from the de Rham theory
   - Optimal order of approximation and number of DOFs
   - We found the direct serendipity space giving AC spaces
3. The construction could be extended to polygonal meshes.