MATH 152 Lab 3

In this lab, we will approximate the value of a particular integral using various types of sums. Use Python to solve each problem.

1. Throughout the assignment, let

$$f(x) = \arccos\left(\frac{1}{1 + 2\cos x}\right)$$

We want to estimate the value of

$$\int_0^{\pi/2} f(x)\, dx$$

(a) Plot $f(x)$ on the interval $x \in \left[0, \frac{\pi}{2}\right]$. (If the plot function is giving you grief, consider using `matplotlib.pyplot`'s plotting function instead of sympy's for a better picture.)

(b) The **Left Endpoint Riemann Sum** is a tool to approximate the value of an integral by summing up the areas of a series of n rectangles of equal width, whose top left corners lie along the function being integrated. The formula is given by

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i \cdot \Delta x$. Throughout this lab, we will use $n = 100$.

i. For the integral given in the problem (and using $n = 100$), compute the value of Δx, and print it out as a floating point number. (Since this lab is all about approximation rather than exact answers, every value you output should be a floating point number!)

ii. Define a list of the n floating point values $f(x_0), f(x_1), f(x_2), \ldots, f(x_{n-1})$. (You don't need to print it out.)

iii. Sum the list and multiply by Δx. The result will be the Left Endpoint Riemann Sum for $\int_0^{\pi/2} f(x)\, dx$. (Make sure you see why!) Thus, it should be a floating point number whose value is fairly close to the actual value of the integral.

2. The **Right Endpoint Riemann Sum** approximates the value of an integral by summing the areas of a series of rectangles whose top *right* corners lie along the function. Thus the formula is

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

Compute the Right Endpoint Riemann sum for $\int_0^{\pi/2} f(x)\, dx$ using $n = 100$. Make sure you see how this is different from the left endpoint sum!

3. The **Midpoint Sum** is similar, but now the midpoints of the top of the rectangles lie along the function. Thus the formula is

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x$$

Compute the Midpoint sum for $\int_0^{\pi/2} f(x)\, dx$ using $n = 100$.

(Problems 4-5 are on the next page.)

4. The **Trapezoid Rule** is a more sophisticated technique that approximates an integral by summing up the areas of a series of n trapezoids. The formula is given by

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$$

   (a) Compute the approximate value of $\int_0^{\pi/2} f(x)\,dx$ using the Trapezoid Rule for $n = 100$. It is possible to do this in Python using the formula above (for example, by writing a list comprehension and adjusting the endpoints as needed), but a simpler approach is to use the `numpy.trapz` function instead.

      i. Define a list X of the $n+1$ floating point values $x_0, x_1, \cdots, x_n$.

      ii. Define a list Y of the $n+1$ floating point values $f(x_0), f(x_1), \cdots, f(x_n)$.

      iii. Call the `trapz` function with the signature `numpy.trapz(Y, X)` to get the Trapezoid Rule approximation for the integral.

   (b) Compute the average of the Left Endpoint and Right Endpoint Riemann Sums (which you found in Problems 1 and 2) for $\int_0^{\pi/2} f(x)\,dx$. Do you notice anything about your answer?

5. **Simpson's Rule** is a much smoother technique that approximates an integral by summing up the areas under a series of quadratic functions. The formula is given by

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 2f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \cdots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$$

   (a) Just like the `trapz` function in Problem 4, there is a `simps` function that uses the same signature and outputs the Simpson's Rule approximation. Thus, use `scipy.integrate.simps(Y, X)` to get the Simpson's Rule approximation for the integral.

   (b) The exact value of the integral, which is found using magic[1], is

$$\int_0^{\pi/2} \arccos\left(\frac{1}{1+2\cos x}\right) = \frac{\pi^2}{6}$$

   For **all five** of the approximations your team found in Problems 1-5, print out the floating point error (that is, the absolute value of the difference) between the approximate value found, and the actual value of the integral.

[1] If you're really curious, see here `https://t1.daumcdn.net/cfile/tistory/1176F84E4F31112B11`