MATH 152 Lab 4

In this lab, you will work various problems involving improper integrals and sequences. Use Python to solve each problem.

1. The **Laplace Transform** of a continuous function $f(t)$ is given by

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

   In this problem, you will calculate the Laplace transforms of several functions using Python's `integrate` function. Remember to define the symbols `s` and `t` using the `positive=True` keyword argument. (Also, remember that ∞ is represented in Sympy as `oo`, and that you can do exponentiation using the `exp` function.)

   (a) $f(t) = 1$
   (b) $f(t) = (t+1)^3$
   (c) $f(t) = t^2 e^{-t}$
   (d) $f(t) = \sin(t)$

2. Consider the integral

$$I_n = \int_0^1 \frac{1-x^n}{1-x}\,dx$$

   where n is a positive integer.

   (a) Plot the function $f(x) = \frac{1-x^4}{1-x}$ in the window $x \in [0,1], y \in [0,5]$.
   (b) Compute the value of the integral I_n for $n=1$, $n=2$, $n=3$, and $n=4$. You may use either exact or approximate integration.
   (c) Compute $I_2 - I_1$, $I_3 - I_2$, and $I_4 - I_3$. What do you think a general form for I_n might be?

3. Consider the related integral

$$J_n = \int_0^1 \frac{x^n - 1}{\ln(x)}\,dx$$

   where n is a positive integer. (Notice the numerator and denominator are both negative when $0 < x < 1$, so the integrand is positive.)

   (a) Plot the function $f(x) = \frac{x^4-1}{\ln(x)}$ in the window $x \in [0,1], y \in [0,5]$.
   (b) Compute the value of the integral J_n for $n=1$, $n=2$, $n=3$, and $n=4$. You may use either exact or approximate integration.
   (c) Compute e^{J_1}, e^{J_2}, and e^{J_3}. What do you think a general form for J_n might be?

4. Define the sequence (a_n) by

$$a_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln(n+1)$$

   where n is a positive integer.

   (a) Compute the values of a_1, a_{10}, a_{100}, a_{1000}, and $a_{1000000}$. You may find it convenient to use a list comprehension to compute the sum in parentheses. Print out all these values as **floating point approximations** (do *not* print them out in exact form.) Does the sequence (a_n) appear to converge or diverge?
   (b) Compute the approximate value of the following improper integral (you may want to do the numerical integration using `mpmath.quad`):

$$\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln(x)}\right)\,dx$$

   Do you notice anything about your answer? (And for an extra challenge, do you see how you might use the results of Problems 2 and 3 to explain this "coincidence"?)