MATH 152 Lab 5

In this lab, you will solve problems concerning infinite series and convergence tests. Use Python to solve each problem.

1. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$
   (a) Find and evaluate an integral that demonstrates that the series converges.
   (b) Determine how many terms of the series we need to compute to ensure that the remainder is less than $\frac{1}{100}$.
   (c) Using the number of terms found in part (b), find (as a floating point number please) the sum of the series to within $\frac{1}{100}$.
   (d) The exact value of the series is
$$\frac{\pi\left(e^{2\pi}+1\right)}{2\left(e^{2\pi}-1\right)} - \frac{1}{2}$$
   Verify that the partial sum you found above is within $\frac{1}{100}$ of this.

2. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$
   (a) Compute the first 10 partial sums of the series. What does it look like the series converges to?
   (b) Plot the first 50 terms of the series and the first 50 partial sums on the same graph.
   (c) Find and evaluate an integral that demonstrates that the series converges.
   (d) Use the `apart` function in Sympy to find the partial fraction decomposition of $\frac{1}{n^2+n}$. Explain (in a print statement) how you might find the exact value of this series using this decomposition.
   (e) Is the sum of this series less than, greater than, or the same as the sum of the series in Problem 1? Does this agree with what you would expect based on comparing these two series to each other?

3. Consider the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$
   By the alternating series test, this series converges.[1]
   (a) Using the remainder test for alternating series, determine how many terms of the series we need to compute to ensure that the remainder is less than $\frac{1}{100}$.
   (b) Using the number of terms found in part (a), find (as a floating point number please) the sum of the series to within $\frac{1}{100}$.

4. Consider the series
$$\sum_{n=1}^{\infty} (8\pi n^2 - 2)e^{-\pi n^2}$$
   (a) Find the first five partial sums of the series in approximate form[2]. What do you think the series converges to?
   (b) *(Extra credit)* For positive values of x, it is known that $e^x > x^2$. Using this fact, it is possible to compare the series to a converging p-series. Specifically, try to find a positive real number c by hand for which
$$\sum_{n=1}^{\infty} (8\pi n^2 - 2)e^{-\pi n^2} \leq \sum_{n=1}^{\infty} \frac{c}{n^2} = \frac{c\pi^2}{6}$$
   Show that your answer to part (a) satisfies this inequality.

[1]The sum of this series is known as Catalan's constant. Its exact value is unknown, and in particular, nobody knows whether it is rational or irrational.
[2]This series converges very quickly, so some of the partial sums may be equal due to floating point round-off.