MATH 152 Lab 6

In this lab, you will solve problems involving power series. Use Python to solve each problem. **Throughout this lab, when defining the symbol n for an index in a power series, make sure you use the keywords `integer=True` and `positive=True`. Otherwise you may get unexpected results!**

1. Consider the power series
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1}$$
   (a) Simplify $\left|\frac{a_{n+1}}{a_n}\right|$ and find the limit.

   (b) State the values of x for which the limit is less than 1 (you may do this by hand or by using `reduce_inequalities` in Python). Your answer will be an interval which is centered at $x = 0$; how far from 0 is each endpoint? (This is called the *radius of convergence* of the series.) Substitute each endpoint into the series to determine whether the series converges or not.

   (c) It can be shown that for values of x where the series converges, it converges to
$$f(x) = \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2 - x + 1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{\pi\sqrt{3}}{18}$$
   (We call these values of x the *interval of convergence* of the series.) To illustrate this, find s_1, s_3, and s_5. Plot these polynomials and f on the same set of axes in its interval of convergence ($y \in [-2, 2]$). If you are getting a math domain error, try shortening the interval of convergence slightly (e.g. subtract something small like 0.01 from the radius).

   (d) The power series
$$\frac{1}{1+t^3} = \sum_{n=0}^{\infty} (-1)^n t^{3n}$$
   is a simple geometric series. Using Python, integrate both $\frac{1}{1+t^3}$ and $(-1)^n t^{3n}$ from $t = 0$ to x. How might you use this result to find the limit of the series given in part (c)?

2. Consider the power series
$$\sum_{n=0}^{\infty} \frac{(2n)!}{n!^2} x^n$$
   (a) Simplify $\left|\frac{a_{n+1}}{a_n}\right|$ and find the limit.

   (b) Find the radius of convergence of this series. Sum the series at each endpoint using the `summation` command to determine whether the series converges or diverges at that endpoint.

   (c) It can be shown that the series converges to $f(x) = \frac{1}{\sqrt{1-4x}}$ on its interval of convergence. To illustrate this, find s_5, s_{10}, and s_{15}. Plot these three polynomials and f on the same set of axes in its interval of convergence ($y \in [0, 5]$).

   (Problem 3 on next page!)

3. Consider the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}x^{-\ln(n)}$$

NOTE: Since the exponents of this series are not integers, the sum of this series isn't mathematically well-defined for negative values of x. So for this problem, you should define the symbol x using `positive=True`.

(a) Simplify $\left|\frac{a_{n+1}}{a_n}\right|$ and find the limit. Does the ratio test tell you anything about this series?

(b) When $0 < x \leq 1$, does $|a_n|$ go to zero? (If it helps, you can try defining $b_n = |a_n|$ with $x = 0.9$ and using Python to find the limit.) What convergence test can you use to determine whether the series converges for $|x| \leq 1$?

(c) When $x > 1$, does $|a_n|$ go to zero? (If it helps, you can try defining $c_n = |a_n|$ with $x = 1.1$ and using Python to find the limit.) What convergence test can you use to determine whether the series converges for $x > 1$?

(d) Based on your answers above, what is the interval of convergence of this series? (Note that since this is not a power series, there's no reason to expect the interval of convergence to be symmetric about any point.)

(e) Compute the sum of the series when x is equal to the constant e. (Do you see how you might be able to use properties of logarithms to find this sum by hand?)