Solutions to MATH 152 Fall 2008 Exam 3B

1. \[ \lim_{n \to \infty} (a_n^2 - 3b_n) = (\lim_{n \to \infty} a_n)^2 - 3 \lim_{n \to \infty} b_n = 2^2 + 3(-3) = -5. \]

2. E Apply L'Hospital’s Rule to \[ \lim_{x \to \infty} \frac{x}{\ln x} = \lim_{x \to \infty} \frac{1}{\frac{1}{x}} = \infty \]

3. E Complete the square: \( (x^2 - 2x + 1) + y^2 + z^2 = 2 + 1; (x - 1)^2 + y^2 + z^2 = 3, \) so \( r^2 = 3 \) and \( r = \sqrt{3}. \)

4. B Using the Comparison Test. (D) is NOT necessarily true because \( \lim_{n \to \infty} a_n = 0 \) does not necessarily mean that \( \sum_{n=1}^{\infty} a_n \) is convergent.

5. A Since \( \lim_{n \to \infty} \frac{n}{n + 1} = 1, \) the sequence \( (-1)^n \frac{n}{n + 1} \) alternates between \(-1 \) and 1, therefore the terms of the series do not approach 0, which means the series diverges by the Test for Divergence.

6. D Let \( a_n = \frac{1}{n} \). Then \( \lim_{n \to \infty} \frac{b_n}{a_n} = 1, \) which means the series \( \sum_{n=1}^{\infty} b_n \) and \( \sum_{n=1}^{\infty} a_n \) either both converge or both diverge. Since \( \sum_{n=1}^{\infty} a_n \) diverges (by Integral Test or P-Test), \( \sum_{n=1}^{\infty} b_n \) diverges by the Limit Comparison Test.

7. B \[ \sum_{n=1}^{\infty} \frac{1}{n(n + 1)(n + 4)} = \sum_{n=1}^{\infty} \left( \frac{1}{n + 3} - \frac{1}{n + 4} \right), \] which is a Telescoping Series:
\[ s_N = \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \left( \frac{1}{6} - \frac{1}{7} \right) + \cdots + \left( \frac{1}{N + 3} - \frac{1}{N + 4} \right) = \frac{1}{4} \] as \( N \to \infty. \)

8. B Since the terms of both series approach zero, both series converge by the Alternating Series Test. To test absolute convergence, we look at \( \sum_{n=1}^{\infty} \frac{1}{n^{1/4}} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^{1/4}}. \) Both are \( P \)-series; in (I), \( P > 1 \) so the series converges, and in (II), \( P < 1 \) so the series diverges. Therefore, series (I) converges absolutely, and series (II) converges but not absolutely.

9. A The series can be written as \[ \sum_{n=1}^{\infty} \left( \frac{2}{9} \right)^{n-1} \left( \frac{2}{3} \right) \], which is a geometric series with \( a = \frac{2}{3}, \) \( r = \frac{2}{3}. \) The sum of the series is \( \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \frac{2}{3}. \)

10. E The Maclaurin series for \( \cos x \) is \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \] So the Maclaurin series for \( \cos(x^2) \) is \[ \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}. \]

11. (a) \( \overrightarrow{AB} = \langle 2, 1, -2 \rangle, \overrightarrow{BC} = \langle 1, 0, 1 \rangle. \) \( \overrightarrow{AB} \cdot \overrightarrow{BC} = (2)(1) + (1)(0) + (-2)(1) = 0, \) so the sides are perpendicular to each other.

(b) \( \overrightarrow{AB} = \sqrt{2^2 + 1^2 + (-2)^2} = 3, \overrightarrow{BC} = \sqrt{1^2 + 0 + 1^2} = \sqrt{2}, \) so the area is \( \frac{1}{2} \overrightarrow{AB} \cdot \overrightarrow{BC} = \frac{3\sqrt{2}}{2}. \)
12. \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \) so \( e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}. \) Therefore, subtracting the first term of the series gives us \( e^{-1} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}. \)

13. \( f(-1) = 0; f'(x) = 8x^3 + 1, \) so \( f'(-1) = -7; f''(x) = 24x^2, \) so \( f''(-1) = 24; f'''(x) = 48x, \) so \( f'''(-1) = -48. \) The third degree Taylor Polynomial is \( f(-1) + \frac{f'(-1)}{1!}(x + 1) + \frac{f''(-1)}{2!}(x + 1)^2 + \frac{f'''(-1)}{3!}(x + 1)^3 = -7(x + 1) + 12(x + 1)^2 - 8(x + 1)^3. \)

14. (a) Applying the Ratio Test gives us absolute convergence when \( \lim_{n \to \infty} \left| \frac{(x-1)^{n+1}}{3^n \sqrt[n]{n}} \right| = \lim_{n \to \infty} \left| \frac{x-1}{3} \right| \frac{1}{\sqrt{n+1}} < 1 \) and divergence when the limit is > 1. Since the second fraction approaches 1, we have absolute convergence when \( \left| \frac{x-1}{3} \right| < 1, \) \( |x-1| < 3 \) which makes the radius of convergence 3.

(b) The series converges when \(-3 < x-1 < 3, -2 < x < 4.\) To find the interval of convergence, test the endpoints: When \( x = -2, \) the series becomes \( \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n \sqrt[n]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}, \) which converges by the Alternating Series test. When \( x = 4, \) the series becomes \( \sum_{n=1}^{\infty} \frac{3^n}{3^n \sqrt[n]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}, \) which diverges by the P-test or integral test. Therefore, the interval of convergence is \(-2 \leq x < 4.\)

15. (a) \( \int S(x) \, dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n} \, dx = \sum_{n=1}^{\infty} \int \frac{(-1)^{n+1}}{(2n+1)!} x^{2n} \, dx = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} x^{2n+1} \)

(b) \( \int_{0}^{1/2} S(x) \, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} \left. x^{2n+1} \right|_{0}^{1/2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} \left( \frac{1}{2} \right)^{2n+1} \)

(c) Since the series is alternating \( |S - S_3| \leq |a_4| = \frac{1}{(2 \cdot 4 + 1)(2 \cdot 4 + 1)!} \left( \frac{1}{2} \right)^{2.4+1} = \frac{1}{9 \cdot 9!} \left( \frac{1}{2} \right)^9 \)