Fall 2009 Math 152

Exam III Version A Solutions

1. **E** \(2a = (2, 4, 6), \ 3b = (9, 6, 3), \) so \(2a - 3b = (-7, -2, 3)\)

2. **A** \(\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{-2 + 3 + 2}{\sqrt{6} \cdot \sqrt{14}} = \frac{3}{\sqrt{84}}\)

3. **D** The dot product must be 0, so we have \(3x + 4 + 2x = 0\), or \(x = -\frac{4}{5}\)

4. **E** Let \(a, b, c\) be the three vectors respectively. The volume is equal to \(|(a \times b) \cdot c|\).

\[
a \times b = \begin{vmatrix}
i & j & k \\
1 & 0 & 1 \\
1 & 2 & 1 \\
\end{vmatrix} = -2i + 2k, \text{ so the volume is } \left|(-2)(0) + (0)(1) + (2)(-1)\right| = 2
\]

5. **A** Complete the square for each quadratic.

\[
x^2 + (y^2 + 2y + 1) + (z^2 + z + \frac{1}{4}) = 1 + 1 + \frac{1}{4} = \frac{9}{4} = r^2, \text{ so } r = \frac{3}{2}.
\]

6. **C** Series (I) is absolutely convergent since \(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\) is a convergent P-series. Series (II) is convergent by the Alternating Series Test, but \(\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}\) is a divergent P-Series, so (II) is divergent, but not absolutely convergent.

7. **A** \(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\). Replace \(x\) with \(-x^2\): \(e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\)

8. **E** \(\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\). Multiply by \(x^2\):

\[
x^2 \sin x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}
\]

9. **D** \(\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\), so \(\cos \pi = -1 = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}\). Multiply by \(\pi\):

\[
\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}.
\]

10. **C** Since the series has a radius of convergence of 3, the series is convergent when \(|x - 3| < 3\) and is divergent when \(|x - 3| > 3\). For series (I), \(|x - 3| = 2(< 3)\) and for series (II), \(|x - 3| = 4(> 3)\). Therefore, (I) is convergent and (II) is divergent.

11. **E** Since \(\sum_{n=1}^{\infty} a_n\) is convergent, we must have \(a_n \to 0\). Therefore, \(\frac{1}{1 + a_n} \to 1\), so the series is divergent by the Test for Divergence.

12. (i) The vector must also be perpendicular to the vectors \(\vec{QP}\) and \(\vec{QR}\) which lie in the plane. \(\vec{QP} = (1, -1, -2)\) and \(\vec{QR} = (1, 0, 0)\). The cross-product is perpendicular to both, so the desired vector is

\[
\begin{vmatrix}
i & j & k \\
1 & -1 & -2 \\
1 & 0 & 0 \\
\end{vmatrix} = -2j + k.
\]

(ii) The area of the triangle is one-half the magnitude of the cross-product found above:

\[
\frac{1}{2}\sqrt{(-2)^2 + 1^2} = \frac{\sqrt{5}}{2}.
\]

13. (i) The series is an alternating series with \(a_n = \frac{1}{n \ln n}\), which is positive, decreasing, and approaching zero. Therefore, the series is convergent by the Alternating Series Test. (ii) To test absolute convergence, look at the series \(\sum_{n=2}^{\infty} \frac{1}{n \ln n}\). Let \(f(x) = \frac{1}{x \ln x}\). \(f\) is positive, continuous, and decreasing. Further, \(\int_{2}^{\infty} \frac{1}{x \ln x} \ dx = \lim_{t \to \infty} \ln (\ln t) - \ln (\ln 2) = \infty\).

Therefore, the series is divergent by the Integral Test, so the original series is NOT absolutely convergent.

14. (i) Apply the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{2^{n+1}(x - 1)^{n+2}}{\sqrt{2n + 3}} - \frac{\sqrt{2n + 1}}{2^n(x - 1)^{n+1}} \right| = \lim_{n \to \infty} \frac{\sqrt{2n + 1} \cdot 2|x - 1|}{\sqrt{2n + 3} \cdot 2^n(x - 1)}.\]

For the series to converge, we need \(2|x - 1| < 1\), or \(|x - 1| < \frac{1}{2}\). Therefore, the radius of convergence is \(\frac{1}{2}\).

(ii) The power series is convergent when \(-\frac{1}{2} < x - 1 < \frac{1}{2}\), or \(\frac{1}{2} < x < \frac{3}{2}\). Test each
endpoint separately. When \( x = \frac{1}{2} \), the series becomes
\[
\sum_{n=0}^{\infty} \frac{2^n (\frac{1}{2} - 1)^n}{\sqrt{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}},
\]
which is convergent by the Alternating Series Test. When \( x = \frac{3}{2} \), the series becomes
\[
\sum_{n=0}^{\infty} \frac{2^n (\frac{1}{2})^n}{\sqrt{2n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2n+1}},
\]
which is divergent by Limit Comparison with \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \).

Therefore, the interval of convergence is \( \left[ \frac{1}{2}, \frac{3}{2} \right) \).

15. \( f(x) = x^5 - 2x^4 + x^3 \) (NOTE: This step is not required, but makes the derivatives much easier to compute).
\[
f(x) = x^5 - 2x^4 + x^3; \quad f(1) = 0
\]
\[
f'(x) = 5x^4 - 8x^3 + 3x^2; \quad f'(1) = 0
\]
\[
f''(x) = 20x^3 - 24x^2 + 6x; \quad f''(1) = 2
\]
\[
f'''(x) = 60x^2 - 48x + 6; \quad f'''(1) = 18
\]
Therefore, \( T_3(x) = \frac{2}{2!} (x-1)^2 + \frac{18}{3!} (x-1)^3 = (x-1)^2 + 3(x-1)^3 \).

16. (i) \( f(x) = \frac{1}{1 + x^4/16} \) which is the sum of a Geometric Series with \( a = \frac{1}{16} \) and \( r = -\frac{x^4}{16} \). Therefore,
\[
f(x) = \sum_{n=0}^{\infty} \left( \frac{x^4}{16} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{16^{n+1}}.
\]
(ii) \( \int_{0}^{1} f(x) \, dx = \int_{0}^{1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{16^{n+1}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)16^{n+1}} \bigg|_{0}^{1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)16^{n+1}}. \)