Exam I Version B Solutions

1. D Integrate by parts with \( u = x^2 \), \( dv = e^x \, dx \). Then \( \int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx \). Integrate by parts again with \( u = 2x \), \( dv = e^x \, dx \) yields \( x^2 e^x - 2xe^x + \int 2e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C \).

2. B Partition along the \( y \)-axis. The slices are square prisms \( 2x \times 2x \times dy = 4x^2 \, dy = 4y \, dy \). Therefore \( V = \int_0^1 4y \, dy = 2y^2 \bigg|_0^1 = 2 \).

3. A The force (weight) of the rope and weight after it has been pulled up \( x \) feet is \( F(x) = 200 - 2x \). Therefore, the work done is given by \( W = \int_{20}^{20} (200 - 2x) \, dx = 200x - x^2 \bigg|_{20}^{20} = 4000 - 400 = 3600 \) ft-lb.

4. B \( \sec^4 x = \sec^2 x \cdot \sec^2 x = (\tan^2 x + 1) \sec^2 x \). Let \( u = \tan x \). Then \( du = \sec^2 x \, dx \).

When \( x = 0 \), \( u = 0 \), and when \( x = \frac{\pi}{4} \), \( u = 1 \). Substituting into the integral yields \( \int_0^1 (u^2 + 1) \, du = \frac{1}{3} u^3 + u \bigg|_0^1 = \frac{4}{3} \).

5. A Let \( u = 10 + x \). Then \( du = dx \).

When \( x = -1 \), \( u = 9 \), and when \( x = 6 \), \( u = 16 \). The integral is therefore equivalent to \( \int_{-1}^{6} \frac{x}{\sqrt{u}} \, du \). Since \( u = 10 + x \), \( x = u - 10 \), so this integral is equivalent to \( \int_{9}^{16} \frac{u - 10}{\sqrt{u}} \, du = \int_{9}^{16} (u^{1/2} - 10u^{-1/2}) \, du \).

6. B The sketch of the region is shown below. Use a partition of the \( y \)-axis, so the functions are \( x = y + 1 \) and \( x = y^2 - 1 \). These functions intersect when \( y + 1 = y^2 - 1, y^2 - y - 2 = 0, (y - 2)(y + 1) = 0 \), or \( y = 2, -1 \). The area is therefore \( \int_{-1}^{2} (y + 1) - (y^2 - 1) \, dy \).

7. E Using the double-angle identity for \( \sin(2x) \) yields \( \int_0^{\pi/2} 2 \sin(x) \cos^2(x) \, dx \). Let \( u = \cos(x) \). Then \( du = -\sin(x) \, dx \).

When \( x = 0 \), \( u = 1 \), and when \( x = \frac{\pi}{2} \), \( u = 0 \). Substituting into the integral yields \( -\int_0^1 2u^2 \, du = \int_0^1 2u^2 \, du = \frac{2}{3} u^3 \bigg|_0^1 = \frac{2}{3} \).

8. C Let \( u = x^2 + 1 \). Then \( du = 2x \, dx \).

When \( x = 0 \), \( u = 1 \), and when \( x = 1 \), \( u = 2 \). Substituting into the integral yields \( \int_1^2 \frac{1}{2} u^{-1/2} \, du = u^{1/2} \bigg|_1^2 = \sqrt{2} - 1 \).

9. D Create cylindrical shells centered around \( x = 2 \). The volume of a shell is \( V = 2\pi rh \, dr = 2\pi(2 - x)(x - x^2) \, dx \).

Since the curves intersect at \( x = 0 \) and \( x = 1 \), the volume of the solid is \( V = 2\pi \int_0^1 (2 - x)(x - x^2) \, dx \).

10. A Integrate by parts with \( u = \ln x \), \( dv = x^3 \, dx \). Then \( du = \frac{1}{x} \, dx \) and \( v = \frac{1}{4} x^4 \).

The integral is equivalent to \( \frac{1}{4} x^4 \ln x - \int \frac{1}{4} x^4 \left( \frac{1}{x} \right) \, dx = \frac{1}{4} x^4 \ln x - \int \frac{1}{4} x^3 \, dx \).
11. The graph is shown below. When \( x = 8, \ y = 16 \) and \( f'(x) = 4 \), so the equation of the tangent line is \( y = 16 + 4(x-8) = 4x - 16 \). It is easiest to partition the \( y \)-axis, so the functions become \( x = 2\sqrt{y} \) and \( x = \frac{1}{4}(y+16) \). The area is given by \( \int_{0}^{16} \left( \frac{1}{4}y + 4 - 2\sqrt{y} \right) dy \).

Integrating yields \( \frac{1}{8}y^2 + 4y - \frac{4}{3}y^{3/2} \bigg|_{0}^{16} = \left( 32 + 64 - \frac{256}{3} \right) = \frac{32}{3} \).

12. 

(a) Let \( u = \tan^{-1}x \) and \( dv = x \ dx \).

Then \( du = \frac{1}{x^2 + 1} \ dx \) and \( v = \frac{1}{2}x^2 \).

Then \( \int x \tan^{-1}x \ dx = \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2} \int \frac{x^2}{x^2 + 1} \ dx \).

\( = \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2} \int \frac{1}{x^2 + 1} \ dx \) = \( \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2} \left( 1 - \frac{1}{x^2 + 1} \right) \) = \( \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2}x - \frac{1}{2}x \tan^{-1}x + C \).

(b) \( \int \cos^2x \tan^3x \ dx = \int \sin^3x \ \cos \ x \ dx \)

\( = \int \frac{(1 - \cos^2x) \sin x}{\cos x} \ dx \). Let \( u = \cos x \). Then \( du = -\sin x \ dx \), so the integral is equivalent to \( -\int \frac{1 - u^2}{u} \ du = -\int \left( \frac{1}{u} - u \right) \ du = -\ln|u| + \frac{1}{2}u^2 + C = -\ln|\cos x| + \frac{1}{2} \cos^2 \ x + C \).

13. A cross-sectional slice a distance \( y \) from the bottom is cylindrical with radius \( x \) and height \( dy \). From similar triangles, \( \frac{2}{3} = \frac{x}{y} \), so \( x = \frac{2}{3}y \). The force (weight) acting on the slice is \( F = \rho g \pi \left( \frac{2}{3}y \right)^2 \ dy \). The distance required to move the slice out of the tank is \( 4 - y \) (including the spout), so the work required to pump the water out of the tank is \( W = \int_{0}^{3} \rho g \pi \left( \frac{4}{9}y^2 (4 - y) \right) dy \) = \( \frac{4\rho g \pi}{9} \int_{0}^{3} (4y^2 - y^3) \ dy = \frac{4\rho g \pi}{9} \left( \frac{4y^3}{3} \right) \bigg|_{0}^{3} = \frac{4\rho g \pi}{9} \left( 36 - \frac{81}{4} \right) = \frac{7\rho g \pi}{2} \) ft-lbs.

14. Slice perpendicular to the \( x \)-axis. The slices are washers with outer radius \( \sec x \), inner radius \( 2 \sin x \), and height \( dx \). Therefore, the volume of the solid is \( V = \pi \int_{0}^{\pi/4} (\sec^2x - 4\sin^2x) \ dx \)

\( = \pi \int_{0}^{\pi/4} (\sec^2x - 2(1 - \cos(2x))) \ dx \)

\( = \pi \left( \tan(x) - 2x + \sin(2x) \right)_{0}^{\pi/4} = \pi \left( 1 - \frac{x}{2} + 1 \right) = 2\pi - \frac{\pi^2}{2} \).

15. The average value is found by calculating \( \frac{1}{b-a} \int_{a}^{b} (3x^2 - 4x - 7) \ dx = \frac{1}{b-a} \int_{0}^{b} (3x^2 - 4x - 7) \ dx = \frac{1}{b-a} \left( x^3 - 2x^2 - 7x \right)_{0}^{b} = b^2 - 2b - 7 \). Set this equal to 1 and solve: \( b^2 - 2b - 7 = 1 \), \( b^2 - 2b - 8 = 0 \), \( (b-4)(b+2) = 0 \), so \( b = 4 \) or \( b = -2 \). Since \( b > 0 \), our only solution is \( b = 4 \).