Exam I Version B Solutions

1. C \[ \int_0^1 \frac{6}{1 + x^2} \, dx = 6 \arctan x \bigg|_0^1 = 6 \left( \frac{\pi}{4} - 0 \right) = \frac{3\pi}{2}. \]

2. E \( \pi \) indicates that the volume is obtained using slicing with a radius of \( \sqrt{x(x-3)^2} = \sqrt{x(x-3)} \). Therefore, this solid is obtained by rotating about the \( x \)-axis the area between \( y = \sqrt{x(x-3)} \) and the \( x \)-axis.

3. B Integrate with respect to \( x \), i.e., Top function - Bottom function and note that these change at \( x = 0 \). The area is \( \int_{-\pi/2}^{\pi/2} \left( \frac{x}{2} - \sin x \right) \, dx + \int_0^a \left( \sin x - \frac{x}{2} \right) \, dx. \)

4. D \( \int \sin^2 x \, dx = \int \frac{1}{2} (1 - \cos 2x) \, dx = \frac{1}{2} \, x - \frac{1}{4} \, \sin 2x, \) so the identity used is \( \sin^2 x = \frac{1}{2} (1 - \cos 2x). \)

5. E Using Hooke’s Law \( F = kx \), we obtain \( 20 = k \left( \frac{1}{3} \right) \), or \( k = 60. \) Then the work required is \( \int_0^{1/2} 60x \, dx = 30x^2 \bigg|_0^{1/2} = \frac{15}{2} \) ft - lbs.

6. C \( T_{avg} = \frac{1}{24 - 0} \int_0^{2\pi} \frac{80 + 10 \sin \left( \frac{\pi}{12} \right)}{24} \, dt = \frac{1}{24} (80t - \frac{120}{\pi} \cos \left( \frac{\pi}{12} \right)) \bigg|_0^{2\pi} = \frac{1}{24} (80 \cdot 24 - \frac{120}{\pi} (\cos 2\pi - \cos 0)) = 80. \) (NOTE that the answer could be obtained without integration by observing that you are averaging over one full period of the cosine function).

7. B Separate the fraction to yield \( \int \left( x^2 - \frac{1}{x} \right) \, dx = \frac{1}{3} x^3 \ln |x| + C. \)

8. E Let \( u = -x^2 \). Then \( du = -2x \, dx. \) When \( x = 0, u = 0, \) and when \( u = 2, x = -\sqrt{2}. \) So the integral becomes \( -\frac{1}{2} \int_{-\sqrt{2}}^0 x^2 e^u \, du. \) Changing boundaries and noting that \( x^2 = -u, \) the integral becomes \( \frac{1}{2} \int_{-\sqrt{2}}^0 -ue^u \, du = -\frac{1}{2} \int_{-\sqrt{2}}^0 ue^u \, du. \)

9. A Let \( y \) be the amount of cable pulled to the top, i.e., the distance the cable has traveled. Then the weight of the cable is given by \( F(y) = 80 - 4y, \) so the work required is \( W = \int_0^{10} (80 - 4y) \, dy = 80y - 2y^2 \bigg|_0^{10} = 600 \) ft - lbs.

10. C Use shells along the \( x \)-axis. Then the radius is \( y = x \) and the height is \( x^3 - 0, \) so the total volume is \( \int_0^1 2\pi x(x^3) \, dx = 2\pi x^5 \bigg|_0^1 = \frac{2\pi}{5}. \)

11. A The answer can be computed, but note that if the average value were \( \frac{1}{2}, \) the area of the rectangle at that height would be equal to the area under the curve. In this graph, the area of the rectangle is larger, so \( \frac{1}{2} (b - a) > \int_a^b f(x) \, dx, \) meaning \( \frac{1}{2} > f_{avg}, \) or \( f_{avg} < \frac{1}{2}. \)

12. D Let \( u = x^3 \) and \( dv = e^{-2x} \, dx. \) Then \( du = 3x^2 \, dx \) and \( v = \frac{1}{2} e^{-2x}. \) Apply the formula \( \int u \, dv = uv - \int v \, du \) to obtain \( -\frac{1}{2} x^3 e^{-2x} + \frac{3}{2} x^2 e^{-2x} \, dx. \)

13. E Convert to sines and cosines: \( \int_{\pi/3}^{\pi/2} \frac{\sin x}{\cos x} \, dx = \int_{\pi/3}^{\pi/2} \frac{\sin x}{\cos x} \, dx = -\cos x \bigg|_{\pi/3}^{\pi/2} = 0 + \frac{1}{2} = \frac{1}{2}. \)

14. Slice along the height of the tank. Then a slice \( y \) units above the bottom is a rectangular prism \( x \times 6 \times dy. \) So the volume is \( V = 6x \, dy. \) Using similar triangles, we obtain \( x = \frac{y}{3} \text{ or } x = \frac{2}{3} y, \) so the volume of the slice is \( 4y \, dy. \) Then the force (weight) is \( 1000g (4y) \, dy. \) Since we define \( y \) from the bottom of the tank, the distance it travels to leave the tank is \( (3 - y) + (spout) = 4 - y. \) At the bottom of the water level, \( y = 0, \) and at the top of the water level (not the tank or spout!) \( y = 2. \) Then the work required to pump all the water out of the tank is \( \int_0^2 1000g (4y)(4 - y) \, dy. \) (NOTE: if you define \( y \) as the distance from the top of the tank, the \( y \) and \( 3 - y \) terms reverse in the solution above to obtain \( \int_1^3 1000g(4)(3 - y)(y + 1) \, dy \).)
15. Slice perpendicular to the \( x \)-axis. Then the cross-
sections are square prisms with volume \((2y)(2y)(dx) = 4y^2 \, dx\). Using the equation of the ellipse we obtain \(y^2 = 16\left(1 - \frac{x^2}{4}\right) = 16 - 4x^2\), so the volume of the solid is \[\int_{-2}^{2} 4 (16 - 4x^2) \, dx = 2 \int_{0}^{2} (64 - 16x^2) \, dx\]

\[
= 2 \left(64x - \frac{16x^3}{3}\right) \bigg|_0^2 = 2 \left(128 - \frac{128}{3}\right) = \frac{512}{3}.
\]

16. The graph is shown below.

Since the bottom function changes, but the left and right functions do not, it is easiest to integrate with respect to \( y \), so the equations of our curves become \( x = y^2 \) (covering both square root curves) and \( x = -2y + 8 \). Equate to find the boundaries: \( y^2 = -2y + 8, y^2 + 2y - 8 = 0\), \((y + 4)(y - 2) = 0\), or \(y = -4\) and \(y = 2\). Then the area is given by

\[
A = \int_{-4}^{2} ((-2y + 8) - y^2) \, dy = -y^2 + 8y - \frac{1}{3}y^3 \bigg|_{-4}^{2} = \left(-4 + 16 - \frac{8}{3}\right) - \left(-16 - 32 + \frac{64}{3}\right) = 36.
\]

17. The graph is shown below.

(a) Since we have functions of \( y \) rotating about the \( y \)-axis, it is easiest to use slices. The volume of the slice is \( \pi R^2 h - \pi r^2 h \), where \( R \) =outer radius = \(4y^2\), \( r \) =inner radius = \(2y^2\), and \( h = dy \). Equate to find the boundaries: \(2y^3 = 4y^2\), \(2y^3 - 4y^2 = 0\), \(2y^2(y - 2) = 0\), so \(y = 0\) and \( y = 2 \). Then the volume is \[\int_{0}^{2} \pi((4y^2)^2 - (2y^3)^2) \, dy.\]

(b) Since we have functions of \( y \) rotating about a line parallel to the \( x \)-axis, it is easiest to use shells. The volume of the shell is \( 2\pi rh \, dr \), where \( r = y - (-2) = y + 2 \), \( dr = dy \), and \( h = 4y^2 - 2y^3 \) (right function - left function). Then the volume of the solid is \[2\pi \int_{0}^{2} (y + 2)(4y^2 - 2y^3) \, dy.\]

18. .

(a) Integrate by parts with \( u = \ln x \) and \( dv = x^{1/4} \, dx \). Then \( du = \frac{1}{x} \, dx \) and \( v = \frac{4}{5} x^{5/4} \). Then

\[
\int x^{1/4} \ln x \, dx = \frac{4}{5} x^{5/4} \ln x - \int \frac{4}{5} x^{5/4} \cdot \frac{1}{x} \, dx = \frac{4}{5} x^{5/4} - \frac{4}{5} \int x^{1/4} \, dx = \frac{4}{5} x^{5/4} - \frac{16}{25} x^{5/4} + C.
\]

(b) Let \( u = 4 - x^3 \). Then \( du = -3x^2 \, dx \). When \( x = 0, u = 4 \), and when \( x = 1, u = 3 \). Substituting these into the integral yields \(-1\int_{0}^{1} x^3 \sqrt{u} \, du\).

Using our original substitution, \( x^3 = 4 - u \) and reversing the boundaries, our integral becomes

\[
\frac{1}{3} \int_{3}^{4} (4 - u)^{3/2} \, du = \frac{1}{3} \int_{0}^{1} (4u^{1/2} - u^{3/2}) \, du = \frac{1}{3} \left(\frac{8}{3} u^{3/2} - \frac{2}{5} u^{5/2}\right) \bigg|_{0}^{1} = \frac{8}{9} (4^{3/2} - \frac{2}{15} (4^{5/2}) - \frac{8}{9} (3^{3/2}) + \frac{2}{15} (3^{5/2}) = \frac{128}{45} - \frac{22}{15} \sqrt{3}.
\]

(c) \(\int_{0}^{\pi/4} \sec^4 x \, dx = \int_{0}^{\pi/4} \sec^2 x(\sec^2 x \, dx)\). Let \( u = \tan x \). Then \( du = \sec^2 x \, dx \). When \( x = 0, u = 0 \), and when \( x = \pi/4, u = 1 \). Use the identity \( \sec^2 x = \tan^2 x + 1 \) to obtain

\[
\int_{0}^{1} (u^2 + 1) \, du = \frac{1}{3} u^3 + u \bigg|_{0}^{1} = \frac{4}{3}.
\]