Exam I Version B Solutions

1. B Let \( u = x \) and \( dv = \cos(4x) \, dx \). Then \( du = dx \) and \( v = \frac{1}{4} \sin(4x) \). Using the Integration by Parts formula, 
\[
\int x \cos(4x) \, dx = \frac{1}{4}x \sin(4x) - \int \frac{1}{4} \sin(4x) \, dx = \frac{1}{4}x \sin(4x) + \frac{1}{16} \cos(4x) + C
\]

2. C Let \( y \) be the amount of cable pulled to the top, i.e., the distance the cable has traveled. Then the weight of the cable is given by \( F(y) = 300 - 2y \), so the work required is 
\[
W = \int_{0}^{90} (300 - 2y) \, dy = 300y - y^2|_{0}^{90} = 27000 - 8100 = 18900 \text{ ft}
\]

3. A Let \( u = 4 + 2x \). Then \( du = 2 \, dx \). When \( x = 0, u = 4 \), and when \( x = 6, u = 16 \). So the integral becomes 
\[
\frac{1}{2} \int_{4}^{16} \frac{x}{\sqrt{u}} \, du
\]
Noting that \( x = \frac{1}{2}(u - 4) \), the integral becomes 
\[
\frac{1}{4} \int_{0}^{16} (u^{1/2} - 4u^{-1/2}) \, du = \int_{0}^{16} \left( \frac{1}{4}u^{1/2} - u^{-1/2} \right) \, du
\]

4. D The graph is shown below.

Since the bottom function changes, but the left and right functions do not, it is easiest to integrate with respect to \( y \), so the equations of our curves become \( x = \frac{1}{2}(y^2 - 8) \) and \( x = y \). Solve to find the boundaries:
\[
\frac{1}{2}(y^2 - 8) = y, \quad y^2 - 2y - 8 = 0, \quad (y - 4)(y + 2) = 0,
\]
or \( y = 4 \) and \( y = -2 \). Then the area is given by 
\[
A = \int_{-2}^{4} \left( y - \frac{1}{2}(y^2 - 8) \right) \, dy
\]

5. B By the Fundamental Theorem of Calculus, 
\[
\int_{0}^{b} f'(x) \, dx = f(b) - f(0) = 3.
\]
Since \( f(4) = 5, f(6) = 0 \).

6. D All answer choices use slicing (about a vertical axis), so slice along the \( y \)-axis. Then the outer radius is \( x = y^{1/2} \), the inner radius is \( x = y^3 \), and the height is \( dy \), so the total volume is 
\[
\int_{0}^{1} \pi((y^{1/2})^2 - (y^3)^2) \, dy = \pi \int_{0}^{1} (y - y^6) \, dy
\]

7. D All answer choices use cylindrical shells (about a vertical axis), so draw shells centered along \( y = 1 \). Then the radius is \( 1 - y \), the height is \( x_{Right} - x_{Left} = y^{1/2} - y^3 \), and the thickness is \( dy \), so the total volume is 
\[
2\pi \int_{0}^{1} (1 - y)(y^{1/2} - y^3) \, dy
\]

8. D Slice perpendicular to the \( x \)-axis from \( x = 0 \) to \( x = 4 \) (the \( x \)-intercept). Then the cross-sections are square prisms with volume \( (y)(y) \, dx \). Using the equation of the line we obtain 
\[
y = \frac{1}{2}(4 - x),\quad \text{so the volume of the solid is}
\]
\[
\int_{0}^{4} \left( \frac{1}{2}(4 - x) \right)^2 \, dx = \int_{0}^{4} (4 - x)^2 \, dx.
\]
Let \( u = 4 - x \), then 
\[
\int_{0}^{4} (4 - x)^2 \, dx = -\frac{1}{3} (4 - x)^3|_{0}^{4} = -\frac{1}{3} (0 - 64) = 16
\]

9. C Let \( u = 25 - x^2 \). Then \( du = -2x \, dx \). When \( x = 0, u = 25 \), and when \( x = \sqrt{5}, u = 0 \). So the integral becomes 
\[
-D \int_{0}^{25} u^{1/2} \, du = \frac{1}{4} \int_{0}^{25} u^{1/2} \, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2}|_{0}^{25} = \frac{1}{6} (25)^{3/2} = \frac{125}{6}
\]

10. A \( f_{avg} = \frac{1}{b - a} \int_{0}^{b} (2 + 10x) \, dx = 4 \), so \( \frac{1}{b} (2x + 5x^{2/3})|_{0}^{4} = 4, \text{ or } 2 + 5b = 4 \). Then \( 5b = 2, \text{ or } b = \frac{2}{5} \).

(Note that the graph of \( f \) is a straight line, so the answer can be obtained geometrically as well, as \( f(x) = f_{avg} \) at \( b = \frac{2}{5} \)).

11. B For the first region \((-1 \leq x \leq 2)\), the area is obtained by integrating \( g(x) - f(x) \). For the second region \((2 \leq x \leq 4)\), the area is obtained by integrating \( f(x) - g(x) \). Therefore, the total shaded area is
\[
\int_{-1}^{2} (g(x) - f(x)) \, dx + \int_{2}^{4} (f(x) - g(x)) \, dx
\]

12. E Using the Fundamental Theorem of Calculus along with the Chain Rule, the derivative is 
\[
F'(x) = \frac{4}{1 + (x^2)^2} \frac{d}{dx} (x^2) = \frac{8x}{1 + x^4}
\]

13. A \( T_{avg} = \frac{1}{\frac{4}{\pi} - 0} \int_{0}^{1/2} (80 + 8 \sin(2\pi t)) \, dt = 2(80t - \frac{4}{\pi} \cos(2\pi t))|_{0}^{1/2} = 2(40 - \frac{4}{\pi} (\cos \pi - \cos 0) = 80 + \frac{16}{\pi}.
\]
14. The graph is shown below.

![Graph of #14B](image)

Although either slices or shells can be used to solve this, to avoid square roots, it is easiest to use slices. The volume of the slice is \( \pi R^2 h - \pi r^2 h \), where \( R \) =outer radius \( = y \), \( r \) =inner radius \( = \frac{y^2}{2} \), and \( h = dy \).

Equate to find the boundaries: \( y = \frac{y^2}{2}, \) \( 2y - y^2 = 0, y(2 - y) = 0 \), so \( y = 0 \) and \( y = 2 \). Then the volume is

\[
\pi \left( \frac{y^2}{2} - \frac{(y^2)^2}{2} \right) dy. = \pi \int_0^2 \left( y^2 - \frac{y^4}{4} \right) dy
\]

\[
= \pi \left( \frac{1}{3}y^3 - \frac{1}{20}y^5 \right)_0^{16\pi} = \frac{16\pi}{15}
\]

15. 

(a) Let \( u = \cos(3t) \). Then \( du = -3\sin(3t) \, dt \).

Substituting these into the integral yields

\[
-\frac{1}{3} \int \sec^2(u) \, du = \frac{1}{3} \tan(u) + C
\]

\[
= \frac{1}{3} \tan(3t) + C.
\]

(b) \( \int_{-1}^1 (x^3 - 4x) \, dx = \frac{1}{4} - 2x^2 \bigg|_{-1}^1 = (4 - 8) - \left( \frac{1}{4} - 2 \right) = \frac{9}{4} \)

(c) Integrate by parts with \( u = \arctan(2x) \) and \( dv = dx \). Then \( du = \frac{2}{1 + 4x^2} \, dx \) and \( v = x \).

Then

\[
\int_0^{\pi/2} \arctan(2x) \, dx = x \arctan(2x) \bigg|_0^{\pi/2} - \int_0^{\pi/2} \frac{2x}{1 + 4x^2} \, dx.
\]

Use substitution on the new integral to obtain

\[
x \arctan(2x) \bigg|_0^{\pi/2} - \frac{1}{4} \ln(1 + 4x^2) \bigg|_0^{\pi/2} = \frac{1}{2} \cdot \pi - \frac{1}{4} \ln(2).
\]

(d) \( \int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x (\sec x \tan x) \, dx \).

Let \( u = \sec x \). Then \( du = \sec x \tan x \, dx \).

Use the identity \( \tan^2 x = \sec^2 x - 1 \) to obtain

\[
\int (u^2 - 1)u^2 \, du = \int (u^4 - u^2) \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C
\]

\[
= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C.
\]

16. Since we have functions of \( x \) and are rotating about a horizontal \( (x) \) axis, we use slices. The volume of the slice is \( \pi R^2 h - \pi r^2 h \), where \( R \) =outer radius \( = \sin(x) - (-1) \), \( r \) =inner radius \( = 0 - (-1) \) (NOTE: this is assuming a lower bound of the \( x \)-axis), and \( h = dx \). Then the volume is

\[
\pi \int_0^{\pi/2} ((\sin x + 1)^2 - (1)^2) \, dx.
\]

17. Slice along the height of the tank. Then a slice \( y \) units above the bottom is a cylindrical disk with volume \( V = \pi x^2 \, dy \). Placing a coordinate system with the origin at the center of the sphere yields the circular equation \( x^2 + y^2 = 5^2 \), so the volume of the slice is \( \pi (25 - y^2) \, dy \). Then the force (weight) is \( 1000 \pi g (25 - y^2) \, dy \). Because of our coordinate system, we let \( y < 0 \) for slices under the center of the sphere, so the distance the slice travels to leave the tank is \( (-1) + 5 \) (radius) \( = 5 - y \). At the bottom of the water level, \( y = -5 \), and at the top of the water level (not the tank or spout!) \( y = 0 \). Then the work required to pump all the water out of the tank is

\[
\int_{-5}^0 1000 \pi g (25 - y^2) (5 - y) \, dy.
\]