MATH 152 Fall 2016
COMMON EXAM III - VERSION A

LAST NAME:  
FIRST NAME:  

INSTRUCTOR:  

SECTION NUMBER:  

UIN:  

DIRECTIONS:

1. The use of a calculator, laptop or cell phone is prohibited.

2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.

3. In Part 1 (Problems 1-20), mark the correct choice on your ScanTron using a No. 2 pencil. The ScanTron will not be returned, therefore for your own records, also record your choices on your exam! Each problem is worth 3 points.

4. In Part 2 (Problems 21-24), present your solutions in the space provided. Show all your work neatly and concisely and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

5. Be sure to write your name, section number and version letter of the exam on the ScanTron form.

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature:  

DO NOT WRITE BELOW!

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PART I: Multiple Choice. 3 points each.

1. Applying the Integral Test to \( \sum_{n=1}^{\infty} \frac{3}{(n+2)(\ln(n+2))^3} \), we find

(a) \( \int_1^{\infty} \frac{3}{(x+2)(\ln(x+2))^3} \, dx = \frac{3}{2[\ln(3)]^2} \), and therefore the series converges.

(b) \( \int_1^{\infty} \frac{3}{(x+2)(\ln(x+2))^3} \, dx = \frac{3}{4[\ln(3)]^4} \), and therefore the series converges.

(c) \( \int_1^{\infty} \frac{3}{(x+2)(\ln(x+2))^3} \, dx = \frac{9}{4[\ln(3)]^4} \), and therefore the series converges.

(d) \( \int_1^{\infty} \frac{3}{(x+2)(\ln(x+2))^3} \, dx = \frac{1}{4[\ln(3)]^4} \), and therefore the series converges.

(e) \( \int_1^{\infty} \frac{3}{(x+2)(\ln(x+2))^3} \, dx = \infty \), and therefore the series diverges.

2. For which values of the fixed positive number \( p \) does \( \sum_{n=1}^{\infty} \frac{n^2}{n^p+1} \) converge?

(a) \( p > 1 \) only
(b) \( p \geq 1 \) only
(c) \( p > 3 \) only
(d) \( p \geq 3 \) only
(e) \( p > 4 \) only

If \( p = 3 \), then \( \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \) acts like \( \sum_{n=1}^{\infty} \frac{1}{n} \) by the Limit Comparison Test, and diverges. \( p \) must be \( > 3 \).

3. The interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n n(x-1)^n}{5^n} \) is

(a) \{1\}
(b) \([-4, 6]\)
(c) \([-4, 6]\)
(d) \([-4, 6]\)
(e) \([-4, 6]\)

When \( x = 6 \): \( \sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n \) diverges by Divergence Test

When \( x = -4 \): \( \sum_{n=1}^{\infty} \frac{(-1)^n (-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n \) diverges by Divergence Test
4. Using the Alternating Series Estimation Theorem, what is the smallest number of terms, \( N \), of the series 
\[
S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}
\] 
that would ensure that the error \( |s - s_N| \) is less than .01? 
\[
\frac{1}{100}
\]
\( b_{N+1} = \frac{1}{(N+1)!} \leq \frac{1}{100} \)
\[
100 < (N+1)!
\]
\( N = 4 \) since \( 5! > 100 \).

5. Using the known Maclaurin series for \( \sin x \), express \( \int \frac{\sin x}{x} \, dx \) as a power series.

(a) \( C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \)
(b) \( C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)} \)
(c) \( C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2} \)
(d) \( C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \)
(e) \( C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \)

6. Suppose it is known that the series \( \sum_{n=1}^{\infty} c_n x^n \) converges at \( x = 3 \) and diverges at \( x = 5 \). Which of the following statements is true?

I. \( \sum_{n=1}^{\infty} c_n x^n \) converges at \( x = -3 \).
II. \( \sum_{n=1}^{\infty} c_n x^n \) diverges at \( x = -6 \).
III. \( \sum_{n=1}^{\infty} c_n x^n \) may converge at \( x = 4 \).

(a) I only.
(b) II and III only.
(c) I and II only.
(d) I, II, and III.
(e) I and III only.
7. Using the Remainder Estimate for the Integral Test, find an upper bound on the error in using $s_n$ to approximate $\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$.

(a) $\frac{2}{20}$  
(b) $\frac{1}{17}$  
(c) $\frac{1}{34}$  
(d) $\frac{2}{17}$  
(e) $\frac{1}{52}$

\[ R_n < \int_4^\infty \frac{x}{(x^2 + 1)^2} \, dx \quad \text{with} \quad u = x^2 + 1 \]
\[ du = 2x \, dx \]
\[ \frac{1}{2} \int \frac{1}{u^2} \, du = \frac{1}{2} \left[ -\frac{1}{u} \right]_4^\infty = 0 + \frac{1}{2(17)} = \frac{1}{34} \]

8. Which of the following is a Maclaurin Series for $f(x) = x \cos(x^3)$?

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n+1)!}$  
(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n)!}$  
(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{(2n)!}$  
(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{(2n)!}$  
(e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

\[ x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!} \]

9. For the third degree Taylor Polynomial, $T_3(x)$, for $f(x) = \ln x$ centered at $x = 2$, what is $T_3(1)$?

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<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(2)$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>$\ln x$</td>
<td>$\ln 2$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{x}$</td>
<td>$\frac{1}{2}$</td>
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<tr>
<td>2</td>
<td>$-\frac{1}{x^2}$</td>
<td>$-\frac{1}{4}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{2}{x^3}$</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
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\[ T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!} (x-2)^2 + \frac{f'''(2)}{3!} (x-2)^3 \]
\[ = \ln 2 + \frac{1}{2} (x-2) - \frac{1}{8} (x-2)^2 + \frac{1}{24} (x-2)^3 \]
\[ T_3(1) = \ln 2 - \frac{1}{2} - \frac{1}{8} - \frac{1}{24} = \ln 2 + \frac{-12-3-1}{24} = \ln 2 - \frac{16}{24} = \ln 2 - \frac{2}{3} \]
10. Find the radius and interval of convergence of the series \( \sum_{n=1}^{\infty} \frac{(x-4)^n}{n!} \).

(a) \( R = 0, I = \{4\} \).
(b) \( R = \infty, I = (-\infty, \infty) \).
(c) \( R = 1, I = (3, 5) \).
(d) \( R = 0, I = \{0\} \).
(e) \( R = 2, I = \{4\} \).

\[
\lim_{n \to \infty} \left| \frac{(x-4)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-4)^n} \right| = \lim_{n \to \infty} \left| \frac{x-4}{n+1} \right| = 0 < 1 \quad \text{for all } x.
\]

Converges for all \( x \).

11. Which of the following tests will determine the convergence of \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5 + 1}} \)?

1. The Comparison Test with \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \)
2. The Comparison Test with \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \)
3. The Comparison Test with \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)

(a) 1
(b) 1 and 2
(c) 2
(d) 2 and 3
(e) none of the above will establish convergence because the series diverges.

12. Suppose \( \ln(3 + 2x) \) is expressed as a power series, centered at zero. What is the radius of convergence?

(a) \( \frac{1}{2} \)
(b) \( \frac{3}{2} \)
(c) \( \frac{3}{2} \)
(d) \( \frac{2}{3} \)
(e) \( \frac{1}{3} \)

\[
\ln \left( 3 \left( 1 + \frac{2}{3}x \right) \right) = \ln 3 + \ln \left( 1 + \frac{2}{3}x \right)
\]

will converge when \( \left| \frac{-2}{3}x \right| < 1 \)

\( \left| x \right| < \frac{3}{2} \)
13. Express \( \int \frac{1}{1+x^5} \, dx \) as a power series.

(a) \( C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1} \)

(b) \( C + \sum_{n=0}^{\infty} \frac{x^{5n+1}}{5^n+1} \)

(c) \( C + \sum_{n=0}^{\infty} \frac{x^{5n+5}}{5n+5} \)

(d) \( C + \sum_{n=0}^{\infty} (-1)^n x^{5n} \)

(e) None of the above

\[
\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n}
\]

\[
\int \sum_{n=0}^{\infty} (-1)^n x^{5n} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1} + C
\]

14. If we find the Taylor Series for \( f(x) = 2e^{2x} \) centered at \( x = 3 \), what is the coefficient of \( (x - 3)^3 \)?

(a) \( \frac{16}{3} e^6 \)

(b) \( \frac{1}{3} e^6 \)

(c) \( \frac{1}{2} e^6 \)

(d) \( \frac{1}{4} e^6 \)

(c) \( \frac{8}{3} e^6 \)

\[
f(x) = 2e^{2x}
\]

\[
f'(x) = 4e^{2x}
\]

\[
f''(x) = 8e^{2x}
\]

\[
f'''(x) = 16e^{2x}
\]

\[
f'''(3) = 16e^6
\]

\[
f'''(3) = \frac{16e^6}{3!} = \frac{16e^6}{6} = \frac{8e^6}{3}
\]

15. Find the Taylor Series for \( f(x) = \frac{1}{x} \) centered at \( a = 2 \).

(a) \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (x - 2)^n \)

(b) \( \sum_{n=0}^{\infty} \frac{(-1)^n+1}{2n+1} (x - 2)^n \)

(c) \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (x - 2)^n \)

(d) \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (x - 2)^n \)

(e) \( \sum_{n=0}^{\infty} \frac{(-1)^n+1}{2n+1} (x - 2)^n \)

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n
\]

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n
\]

\[
[\text{Is there a typographical error in the last line? It seems repeated.}]
\]
16. Which of the following series DIVERGES?

- (a) \( \sum_{n=2}^{\infty} \frac{n!}{2^n} \)  
  - Ratio Test gives limit of \( 0 \) 
- (b) \( \sum_{n=1}^{\infty} \frac{1}{2^n} \)  
  - Ratio Test gives limit of \( \frac{1}{2} \) 
- (c) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)  
  - Converges by AST. 
- (d) \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \)  
  - Converges by Comparison or Limit Comparison with \( \frac{1}{n^2} \).
- (e) \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+5} \)  
  - Converges by AST.

17. \( \sum_{n=0}^{\infty} \frac{(-4)^n}{n!} = e^{-4} \)  
   - since \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \)

18. Which of the following is a Maclaurin series for \( f(x) = e^{-x^3} \)?

- (a) \( \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \)
- (b) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} \)
- (c) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(3n)!} \)
- (d) \( \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \)
- (e) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} \)
19. Which of the following statements is true?

(a) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}} \) is a convergent p-series. \( p = \frac{2}{3} < 1 \). Diverges.
(b) \( \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1} \) diverges by the Test for Divergence. \( \lim_{n \to \infty} \frac{n^2}{n^3 + 1} = 0 \)
(c) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \) is a convergent series. \( \checkmark \) by AST
(d) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \) converges but not absolutely. Both \( \sum \frac{(-1)^n}{n^2} \) and \( \sum \frac{1}{n^2} \) converge.
(e) None of the above statements are true.

20. If we approximate \( f(x) = \sin x \) with a third degree Taylor polynomial centered at \( x = \frac{\pi}{3} \), use Taylor's Inequality to estimate the accuracy of the approximation \( \sin x \approx T_3(x) \) for \( x \) in the interval \( \left[ 0, \frac{2\pi}{3} \right] \).

**Taylor's Inequality:** If \( T_n(x) \) is used to approximate \( f(x) \) centered at \( x = a \), an upper bound on the absolute value of the remainder is \( |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \), where \( x \) is in an interval containing \( a \).

(a) \( |R_3(x)| \leq \frac{\sqrt{3} (\pi/3)^3}{12} \)
(b) \( |R_3(x)| \leq \frac{\sqrt{3} (\pi/3)^4}{48} \)
(c) \( |R_3(x)| \leq \frac{(\pi/3)^4}{24} \)
(d) \( |R_3(x)| \leq \frac{(\pi/3)^4}{48} \)
(e) \( |R_3(x)| \leq \frac{(\pi/3)^3}{5} \)

\( f(x) = \sin x \)
\( f'(x) = \cos x \)
\( f''(x) = -\sin x \)
\( f'''(x) = -\cos x \)
\( f^{(4)}(x) = \sin x \)

\( M \) is an upper bound on \( |f^{(n+1)}(x)| \) on interval \( |x-a|^{n+1} \), \( M = 1 \)

Now, on interval \( \left[ 0, \frac{2\pi}{3} \right] \), upper bound for \( |x-\frac{\pi}{3}| \) is \( \frac{\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3} \), and \( \frac{\pi}{3} \leq \frac{2\pi}{3} \)

\[ S_3 \left| R_3(x) \right| \leq \frac{1}{4!} \left( \frac{\pi}{3} \right)^4 = \frac{1}{24} \left( \frac{\pi}{3} \right)^4 \]
PART II: Work Out: Box your final answer

21. Show the following series converge or prove they diverge.
   a.) (4 pts) \( \sum_{n=1}^{\infty} \frac{4 + \arctan(n)}{n^2} \leq \sum_{n=1}^{\infty} \frac{4 + \sqrt{2}}{n^2} \)

   \( \sum_{n=1}^{\infty} \frac{4 + \sqrt{2}}{n^2} \) is a convergent p-series. \((p = 7 > 1)\)

   By the Comparison Test, since our series is smaller than a convergent series, it also converges.

   b.) (4 pts) \( \sum_{n=1}^{\infty} \frac{\sqrt{n^3} - 3n}{n^4 + 4n^2 + 3} \) Consider the series \( \sum_{n=3}^{\infty} \frac{4n^3}{n^4} = \sum_{n=3}^{\infty} \frac{4}{n} \),

   which is a divergent p-series.

   \[ \lim_{n \to \infty} \frac{4n^3 - 3n}{n^4 + 4n^2 + 3} = \lim_{n \to \infty} \frac{4n^3 - 3n}{n^4 + 4n^2 + 3} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{4n - 3}{n^4 + 4n^2 + 3} = 1 \]

   By LCT, both series do the same thing. So, our series diverges.

   c.) (4 pts) \( \sum_{n=0}^{\infty} \frac{(-1)^{n}(n-1)}{n+2} \)

   \[ \lim_{n \to \infty} \frac{n-1}{n+2} = 1 \text{ and so } \lim_{n \to \infty} \frac{(-1)^{n}(n-1)}{n+2} \neq 0 \]

   Diverges by Test for Divergence.
22. (8 pts) Determine whether the series converges absolutely, converges but not absolutely or diverges: \( \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{a \ln(n)} \).

The series \( \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln n} \) converges by AST since:

a) \( \lim_{n \to \infty} \frac{1}{n \ln n} = 0 \) and

b) \( \frac{1}{n \ln n} \) is decreasing. \([\frac{1}{n \ln n} > \frac{1}{(n+1) \ln(n+1)}]\)

Consider the series \( \sum_{n=2}^{\infty} \left| \frac{(-1)^{n-1}}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \)

\( \int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \ln(\ln x) \bigg|_{2}^{\infty} = \infty \). Diverges, so series diverges.

Therefore series converged, but not absolutely.

23. (10 pts) Express \( f(x) = \frac{4x}{(x^2 + 4)^2} \) as a power series centered at zero. Include the radius of convergence in your answer.

\( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \)

Differentiably, we get \( \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \) or \( \sum_{n=0}^{\infty} (n+1)x^n \)

Now, \( \frac{1}{(4+x^2)^2} = \frac{1}{(4(1+x^2/4))^2} = \frac{1}{16} - \frac{1}{(1+x^2/4)^2} = \frac{1}{16} \cdot \frac{1}{(1-(x^2/4))^2} \)

\( = \frac{1}{16} \cdot \sum_{n=0}^{\infty} (n+1) \left(\frac{x^2}{4}\right)^n = \frac{1}{16} \sum_{n=0}^{\infty} (n+1)(-1)^n \frac{x^{2n}}{4^n} \)

\( = \sum_{n=0}^{\infty} \frac{(n+1)(-1)^n x^{2n}}{4^{n+2}} \)

Finally, \( \frac{4x}{(x^2 + 4)^2} = \sum_{n=0}^{\infty} \frac{4(n+1)(-1)^n x^{2n+1}}{4^{n+2}} = \sum_{n=0}^{\infty} \frac{(n+1)(-1)^n x^{2n+1}}{4^{n+1}} \)

Need \( \left| \frac{x^2}{4} \right| < 1 \Rightarrow |x^2| < 4 \Rightarrow |x| < 2 \ (R = 2) \)
24. (10 pts) Find the interval of convergence of the series \( \sum_{n=2}^{\infty} \frac{(-2)^n(x+5)^n}{\sqrt{n+1}} \). Be sure to test the endpoints for convergence.

\[
\lim_{n \to \infty} \left| \frac{(-2)^n(x+5)^n}{\sqrt{n+2}} \right| = \lim_{n \to \infty} \left| -\frac{2^{n+1}(x+5)}{\sqrt{n+2}} \right|
\]

\[
= \left| -2(x+5) \right| = \left| 2(x+5) \right| < 1
\]

\[
|x+5| < \frac{1}{2}
\]

\[
R = \frac{1}{2}
\]

When \( x = -\frac{11}{2} \):

\[
\sum_{n=2}^{\infty} \frac{(-2)^n(-\frac{11}{2})^n}{\sqrt{n+1}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n+1}} \text{ Diverges by LCT with } \sum \frac{1}{\sqrt{n}}.
\]

When \( x = -\frac{9}{2} \):

\[
\sum_{n=2}^{\infty} \frac{(-2)^n(\frac{1}{2})^n}{\sqrt{n+1}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \rightarrow \text{ Converges by AST.}
\]

\[
\therefore \quad \left( -\frac{11}{2}, -\frac{9}{2} \right]
\]