DIRECTIONS:

1. The use of a calculator, laptop or computer is prohibited.

2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.

3. In Part 1, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore for your own records, also record your choices on your exam!

4. In Part 2, present your solutions in the space provided. Show all your work neatly and concisely and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

5. Be sure to fill in your name, UIN, section number and version letter of the exam on the ScanTron form.

THE AGGIE CODE OF HONOR

“An Aggie does not lie, cheat or steal, or tolerate those who do.”

Signature: ________________________________
PART I: Multiple Choice. 4 points each

1. Compute \( \int_{0}^{\sqrt{\pi}} x \sin(\pi - x^2) \, dx \).

A. \( -\sin \sqrt{\pi} \)
B. \( -1 \)
C. \( -2 \)
D. 2
E. 1

Set \( u = \pi - x^2 \), then \( du = -2x \, dx \). Then the integral becomes:

\[
\int_{\pi}^{0} x \sin(u) \, du = \frac{1}{2} \int_{0}^{\pi} \sin(u) \, du = 1.
\]

2. Compute \( \int_{0}^{1} xe^{2x} \, dx \).

A. \( \frac{e^2 + 1}{4} \)
B. \( \frac{3e^2 + 1}{4} \)
C. \( -2e^2 \)
D. \( \frac{e + 1}{4} \)
E. \( 6e^2 \)

This is integration by parts. We begin with \( u = x \), and \( dv = e^{2x} \, dx \) and so \( du = dx \) and \( v = \frac{1}{2}e^{2x} \). So that:

\[
\int_{0}^{1} xe^{2x} \, dx = \frac{1}{2}e^{2x} \bigg|_{0}^{1} - \int_{0}^{1} \frac{1}{2}e^{2x} \, dx
\]

\[
= \frac{1}{2}e^2 - \frac{1}{4}(e^2 - 1)
\]

\[
= \frac{e^2 + 1}{4}.
\]

3. An ideal spring (i.e. it obeys Hooke’s Law) has a natural length of 10 meters. The work done in stretching the spring from 14 meters to 18 meters is 24J. Determine the spring constant.

A. \( k = \frac{1}{2} \text{N/m} \)
B. \( k = \frac{3}{8} \text{N/m} \)
C. \( k = \frac{1}{2} \text{N/m} \)
D. \( k = \frac{3}{4} \text{N/m} \)
E. \( k = \frac{3}{14} \text{N/m} \)

The work done is \( W = 24 = \int_{4}^{8} kx \, dx = k \frac{1}{2}(64 - 16) = k24 \). So then \( k = 1 \).
4. Which of the following integrals gives the area of the region bounded by the curves \( y = \frac{1}{x} \), \( y = \frac{x - 1}{e(e - 1)} \) and \( x = 1 \). (Hint: the curves \( y = \frac{1}{x} \) and \( y = \frac{x - 1}{e(e - 1)} \) intersect at the point \( (e, \frac{1}{e}) \)).

A. \( \int_{1/e}^{1} \frac{dy}{y} + \int_{0}^{e} (e(e - 1)y + 1)dy \).

B. \( \int_{1}^{e} \left( \frac{1}{x} - \frac{x - 1}{e(e - 1)} \right)dx \).

C. \( \int_{0}^{e} \left( \frac{1}{x} - \frac{x - 1}{e(e - 1)} \right)dx \).

D. \( \int_{1}^{e} \left( \frac{x - 1}{e(e - 1)} - \frac{1}{x} \right)dx \).

E. \( \int_{0}^{1} (\frac{1}{y} - 1)dy \).

The formula is integral of bigger minus integral of smaller.
You’ve been given the intersection points and so the integral is:
\[ \int_{1}^{e} \left( \frac{1}{x} - \frac{x - 1}{e(e - 1)} \right)dx. \]

5. Find the volume of the solid obtained by rotating the region bounded by \( y = x^2 \) and \( y = x^3 \) (in the first quadrant) around the \( x \)-axis.

A. \( \frac{2}{35} \).

B. \( \pi \frac{1}{12} \).

C. \( \frac{1}{12} \).

D. \( \pi \frac{2}{35} \).

E. \( \pi \frac{1}{105} \).

This is a “washers problem”. The black lines roughly illustrate one such washer.
The volume of this washer is \( \Delta V = \pi(R^2 - r^2) \Delta x \). Here \( R = x^2 \) and \( r = x^3 \).
So then the integral is:
\[ V = \int_{0}^{1} \pi(x^4 - x^6)dx \]
\[ = \pi \left( \frac{1}{5} - \frac{1}{7} \right) \]
\[ = \frac{2\pi}{35}. \]
6. Which of the following is equal to \( \int_{0}^{1} x^2 \sin(x-1) \, dx \)?

A. \( \int_{-1}^{0} u^2 \sin(u) \, du \)

B. \( \int_{0}^{1} x^2 \sin(u) \, du \)

C. \( \int_{-1}^{0} (u+1)^2 \sin(u) \, du \)

D. \( \int_{0}^{1} u^2 \sin(u) \, du \)

E. \( \int_{0}^{1} (u+1)^2 \sin(u) \, du \)

This is u-substitution. Take \( u = x - 1 \), then \( du = dx \). Looking ahead, we will need to compute \( x^2 = (u+1)^2 \). With these substitutions, the integrals is:

\[
\int_{-1}^{0} x^2 \sin u \, du = \int_{-1}^{0} (u+1)^2 \sin u \, du.
\]

7. Find the area bounded by the curves \( y = 2 - x^2 \) and \( y = x^2 \).

The top graph is \( y = 2 - x^2 \) and the bottom is \( y = x^2 \). The intersection points are found by solving \( 2 - x^2 = x^2 \) which is \( 2 = 2x^2 \). That is, \( x = \pm 1 \). Then the intersection points are \((-1, 1) \) and \((1, 1) \). So the integral to compute is:

\[
\int_{-1}^{1} (2 - x^2) - x^2 \, dx = 2 \int_{-1}^{1} (1 - x^2) \, dx
\]

\[
= 2 \left( 2 - \frac{2}{3} \right)
\]

\[
= \frac{8}{3}.
\]
8. A 90ft cable weighing 10 lb is hanging down the side of a 200 ft building. There is a 50 lb bucket of rocks attached to the rope. How much work is required to pull the rope with the bucket of rocks 30 feet up the side of the building?

First, the work for the rocks is just \( W_{\text{rocks}} = (50)(30) = 1500 \). We need to break the rope into two pieces. There is the part of the rope that goes over the top of the building, and the part of the rope that stays hanging. For the part that stays hanging, the work is: \( W_{\text{rope}_1} = \text{(weight of rope)}(30) = 200 \). Finally, the work for the other part of the rope is found by breaking the rope into small pieces. The work done to each small piece is \( \frac{1}{9} \Delta y(90 - y) \). Only the top 30 feet make it over the top of the roof, and so the work done on this part of the rope is:

\[
W_{\text{rope}_2} = \int_{60}^{90} \frac{1}{9}(90 - y)dy = -\frac{1}{9} \int_{30}^{0} udu = \frac{1}{2} \cdot 900 = 50.
\]

Thus, the total work is 1500 + 200 + 50 = 1750.

9. Which of the following is equal to \( \int_{\pi/4}^{\pi} \tan^2(\theta) \sec^4(\theta)d\theta \)?

A. \( \int_{0}^{1} u^2(1 - u^2)du \).

Recall that \( \sec^2(\theta) = 1 + \tan^2(\theta) \) and \( \frac{d\tan \theta}{d\theta} = \sec^2(\theta) \). Writting the integral as \( \int_{0}^{\pi} \tan^2(\theta) \sec^4(\theta)d\theta = \int_{0}^{\pi} \tan^2(\theta)(1 + \tan^2(\theta)) \sec^2(\theta)d\theta \), we see to choose \( u = \tan(\theta) \). Then \( du = \sec^2(\theta)d\theta \) and so:

\[
\int_{0}^{\pi} \tan^2(\theta) \sec^4(\theta)d\theta = \int_{0}^{1} u^2(1 + u^2) \sec^2(\theta) \frac{du}{\sec^2\theta} = \int_{0}^{1} u^2(1 + u^2)du.
\]

B. \( \int_{0}^{\pi} u^2(1 + u^2)du \).

C. \( -\int_{0}^{1} u^2(1 + u^2)du \).

D. \( \int_{0}^{1} u^2(1 + u^2)du \).

E. \( \int_{0}^{\pi} u^2(1 - u^2)du \).

10. Compute \( \int_{1}^{4} \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \).

A. -6.

B. \( 2e - 2 \).

This is a \( u \)-sub problem. Let \( u = \sqrt{x} \), then \( du = \frac{1}{2\sqrt{x}} \). The integral becomes:

\[
\int_{1}^{4} \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int_{1}^{2} u^{2} e^{u} du = 2 \int_{1}^{2} e^{u} du = 2(e^2 - e).
\]

C. \( -2e^2 + 2e \).

D. 6.

E. \( 2e^2 - 2e \).
11. Find the area between \( y = \cos x \) and \( y = \sin x \) from \( x = 0 \) to \( x = \frac{5\pi}{4} \).

\[
\begin{align*}
\text{A. } & 3\sqrt{2} - 1 \\
\text{B. } & 1 + \sqrt{2} \\
\text{C. } & 1 - 3\sqrt{2} \\
\text{D. } & \sqrt{2} - 1 \\
\text{E. } & (\sqrt{2} - 1) - 2\sqrt{2}
\end{align*}
\]

Notice that we will have to split this into two integrals. Also, the two curves intersect at \( \left( \frac{\pi}{4}, \frac{1}{\sqrt{2}} \right) \) and \( \left( \frac{5\pi}{4}, -\frac{1}{\sqrt{2}} \right) \). So, the area is:

\[
\int_0^{\frac{\pi}{4}} (\cos x - \sin x) \, dx + \int_{\frac{5\pi}{4}}^{\pi} (\sin x - \cos x) \, dx
\]

\[
= \frac{6}{\sqrt{2}} - 1 = 3\sqrt{2} - 1.
\]

12. Which integral gives the volume of the solid obtained by rotating the region bounded by \( y = \sin x \) and \( y = \frac{2}{\pi} x \) in the first quadrant around the \( y \)-axis?

\[
\begin{align*}
\text{A. } & 2\pi \int_0^{\frac{\pi}{2}} x(\sin x - \frac{2}{\pi} x) \, dx. \\
\text{B. } & 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\sin x - \frac{2}{\pi} x}{2} \right) \, dx. \\
\text{C. } & 2\pi \int_0^{\frac{\pi}{2}} x(\sin x - \frac{2}{\pi} x) \, dx. \\
\text{D. } & \pi \int_0^{\frac{\pi}{2}} (\sin^2 x - \frac{4}{\pi^2} x^2) \, dx. \\
\text{E. } & \pi \int_0^{\frac{\pi}{2}} x(\sin x - \frac{2}{\pi} x) \, dx.
\end{align*}
\]

This is a “cylindrical shells” problem. The heights of the shells are \( \sin x - \frac{2}{\pi} x \) and the radii are \( x \).

\[
2\pi \int_0^{\frac{\pi}{2}} x(\sin x - \frac{2}{\pi} x) \, dx.
\]
13. Compute $\int_1^e \ln(x^2)\,dx$.
   A. $2e - 1$.
   B. $4e - 2$.
   C. $1$.
   D. $\frac{2(1 - e)}{e}$.
   E. $2$.

   First, $\ln x^2 = 2 \ln x$. So the integral is $2 \int_1^e \ln x\,dx$. This is an integration by parts problem. We take $u = \ln x$ and $dv = 1$. Then $du = \frac{1}{x}$ and $v = x$. So the integral is:
   
   $$2 \int_1^e \ln x\,dx = 2 \left( x \ln x \Big|_1^e - \int_1^e du \right) = 2(e - (e - 1)) = 2.$$

14. Which integral gives the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 0$, and $x = 1$ about the line $x = 2$?
   A. $2\pi \int_0^1 x^3(1 + x)\,dx$.
   B. $2\pi \int_0^1 x^3(2 - x)\,dx$.
   C. $\pi \int_0^1 (2 - y^{\frac{1}{3}})^2 dy$.
   D. $2\pi \int_0^2 x^3(2 - x)\,dx$.
   E. $2\pi \int_0^2 x^3(1 + x)\,dx$.

   This is another cylindrical shells problem (actually, you can do it with washers, too). The height of the shells are $x^3$. The radius is $2 - x$ (since the axis is $x = 2$). So the integral is:
   
   $$2\pi \int_0^1 x^3(2 - x)\,dx.$$

15. Compute $\int_0^{\pi/2} \cos^2(\theta) \sin^3(\theta) d\theta$.
   A. $\frac{\pi^2}{30}$.
   B. $\frac{2}{15}$.
   C. $-\frac{2}{15}$.
   D. $0$.
   E. $-\frac{\pi^2}{30}$.

   The first step is to write $\sin^2 \theta = \sin^2 \theta \sin \theta = (1 - \cos^2 \theta) \sin \theta$. Then we choose $u = \cos \theta$ and $du = -\sin \theta d\theta$. And so the integral becomes:

   $$\int_0^{\pi/2} \frac{1}{2} \cos^2(\theta) \sin^3(\theta) d\theta = \int_0^0 u^2(1 - u^2) \sin \theta \frac{du}{-\sin \theta}$$
   $$= \int_0^1 u^2(1 - u^2) du = \frac{2}{15}.$$
16. Which of the following is equal to \( \int_{0}^{\pi} x^2 \cos x \, dx \)? (Hint: just do integration by parts once.)

A. \( \pi - 2 \int_{0}^{\pi} x \sin x \, dx \).

B. \( \pi + 2 \int_{0}^{\pi} x \sin x \, dx \).

C. \( -2 \int_{0}^{\pi} x \sin x \, dx \).

D. \( -\pi + 2 \int_{0}^{\pi} x \sin x \, dx \).

E. \( 2 \int_{0}^{\pi} x \sin x \, dx \).

For this problem, take \( u = x^2 \), \( dv = \cos x \, dx \). Then \( du = 2x \, dx \) and \( v = \sin x \). So, the integral is:

\[
\int_{0}^{\pi} x^2 \cos x \, dx = x^2 \sin x |_{0}^{\pi} - \int_{0}^{\pi} 2x \sin x \, dx = -2 \int_{0}^{\pi} \sin x \, dx.
\]

**PART II WORK OUT**

**Directions**: Present your solutions in the space provided. Show all your work neatly and concisely and box your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

17. (9 points) Compute \( \int \cos^2 x \sin^2 x \, dx \).

First, note that \( \cos x \sin x = \frac{1}{2} \sin(2x) \) so that \( \cos^2 x \sin^2 x = \left( \frac{1}{2} \sin(2x) \right)^2 = \frac{1}{4} \sin^2(2x) \). Also, \( \sin^2(2x) = \frac{1 - \cos 4x}{2} \) So the integral is:

\[
\int \cos^2 x \sin^2 x \, dx = \frac{1}{4} \int \sin^2(2x) \, dx = \frac{1}{4} \int \frac{1 - \cos 4x}{2} \, dx = \frac{1}{8} \int (1 - \cos 4x) \, dx = \frac{4x - \sin 4x}{32} + C.
\]
18. (9 points) Compute $\int e^{2x} \sin x \, dx$.

You will probably recognize this as a “wrap around” problem. We start by doing integration by parts. We choose $u = \sin x$ and $dv = e^{2x} \, dx$. Then $du = \cos x \, dx$ and $v = \frac{1}{2}e^{2x}$ and the integral becomes:

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx.$$  

For the integral that is left, we to integration by parts again. We choose $u = \cos x$, $dv = e^{2x} \, dx$. Then $du = -\sin x \, dx$ and $v = \frac{1}{2}e^{2x}$. Thus, the above becomes:

$$\int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{2} \left( \frac{1}{2}e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x \, dx \right)$$

$$= \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x \, dx.$$  

Rearranging, this gives:

$$\frac{5}{4} \int e^{2x} \sin x \, dx = \frac{1}{2}e^{2x} \sin x - \frac{1}{4}e^{2x} \cos x,$$

and so,

$$\int e^{2x} \sin x \, dx = \frac{2}{5}e^{2x} \sin x - \frac{1}{5}e^{2x} \cos x + C.$$
19. (9 points) Find the volume of the solid whose base is the circle \( x^2 + y^2 = 1 \) and whose cross sections perpendicular to the \( y \) axis are equilateral triangles. (You may find it helpful to know that the area of an equilateral triangle is \( \frac{\sqrt{3}}{4} \ell^2 \) where \( \ell \) is the sidelength.)

The volume of a small triangle is

\[
\Delta V = \text{Area} \times \Delta \text{Width} = \left( \frac{\sqrt{3}}{4} \ell^2 \right) \Delta y.
\]

Here \( \ell \) is the width of the base, which is \( 2x \). Since \( x \) lies on the circle \( x^2 + y^2 = 1 \), and so \( \ell^2 = (2x)^2 = 4(1 - y^2) \). So:

\[
\left( \frac{\sqrt{3}}{4} \ell^2 \right) \Delta y = \sqrt{3}(1 - y^2) \Delta y.
\]

So, the volume is:

\[
V = \sqrt{3} \int_{-1}^{1} (1 - y^2) \, dy = 2\sqrt{3} \int_{0}^{1} (1 - y^2) \, dy = \frac{4}{\sqrt{3}}.
\]
20. (9 points) A hemispherical tank of radius 2 ft is filled with a foot of a liquid with weight density \( \rho g \ \text{lb/ft}^3 \). There is a one foot spout mounted at the top of the tank through which water is drained. Set up the integral that gives the work required to pump the water out of the tank through the spout. **Indicate on the picture where you are placing the axes and which direction is positive.** DO NOT INTEGRATE.

The weight of a small slice of water is \( \rho g \Delta \text{Vol} \). The volume of a small slice of water is \( \pi x^2 \Delta y \). Since \( x \) lies on the circle \( x^2 + y^2 = 4 \), this is \( \pi (4 - y^2) \Delta y \). The distance from this slice to the top of the tank is \( 0 - y = -y \) and the distance from the top of the tank to the spout is 1. Thus, the overall distance the water travels is \( 1 - y \). So, the work done on this small slice of liquid is:

\[
\Delta W = \text{distance} \rho g \Delta \text{Vol} = (1 - y)\pi(4 - y^2) \Delta y.
\]

So the total work is:

\[
W = \pi \rho g \int_{-2}^{-1} (1 - y)(4 - y^2) \, dy = \pi \rho g \int_{1}^{2} (1 + y)(4 - y^2) \, dy.
\]
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