Algebra Qualifying Examination
9 January 2024

Instructions:

- There are nine questions worth a total of 100 points.
- **Read all problems first**. Make sure that you understand each and feel free to ask clarifying questions. Do not interpret a problem in a way that makes it trivial.
- Credit is awarded based both on the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner and be legible. State clearly any major theorems you use (hypotheses and conclusions). Justify your reasoning.
- Start each problem on a new page, clearly marking the problem number and your name on that page. Do 'scratch work' on a separate page.
- Rings always have an identity (otherwise they are rng) and all R-modules are left modules.

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1. [12 pts] Let $n \in \mathbb{N}$ and let $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{Q})$ be the matrix at right where $a_{ij} = i$ for $1 \leq i, j \leq n$. Determine, with justification, the following:

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & n \end{pmatrix}$$

   (a) The characteristic polynomial of A.
   (b) The minimal polynomial of A.
   (c) The Jordan canonical form of A.
   (d) The rational canonical form of A.

2. [10 pts] (1) Let H be a Sylow p-subgroup of a finite group G and let K be a subgroup of G. Is it always the case that $H \cap K$ is a Sylow p-subgroup of K? Justify your answer.
   (2) Show that a group of order $2024 = 2^3 \cdot 11 \cdot 23$ cannot be simple.
3. [10 pts] Let M and N be normal solvable subgroups of a group G. Show that their product MN is also solvable.
4. [10 pts] Let $\mathbb{K}$ be a field and let $R = \mathrm{Mat}_n(\mathbb{K})$ be the ring of $n \times n$-matrices with entries from $\mathbb{K}$. Let $f\colon R \to S$ be any ring homomorphism. Show that f is either injective or zero.
5. [12 pts] Let R be an integral domain with field of fractions $\mathbb{K}$. For a non-zero ideal $I \subseteq R$, set
$$I^{-1} := \{\beta \in \mathbb{K} \mid \beta I \subseteq R\}.$$
   (a) Prove that I^{-1} is an R-submodule of $\mathbb{K}$ and that $I^{-1}I \subseteq R$.
   (b) The ideal I is *invertible* if $I^{-1}I = R$. Prove the following statement: I is invertible if and only if there exist $a_1, \ldots, a_n \in I$ and $q_1, \ldots q_n \in \mathbb{K}$ so that
      (i) $q_i I \subseteq R$ for all $i = 1, \ldots, n$, and
      (ii) $1 = \sum_{i=1}^n q_i a_i$.
6. [10 pts] Let $\mathbb{F}$ be a field, $R = \mathbb{F}[x]$, the polynomial ring, and $M = \mathbb{F}[[X]]$, the ring of formal power series. Show that M is not a free R-module.
7. [10 pts] Let R be a commutative ring and $0 \to M \xrightarrow{f} N$ be an exact sequence of R-modules (f is injective). Let P be a projective R-module. Show that $M \otimes_R P \xrightarrow{f \otimes 1_P} N \otimes_R P$ is injective.

Two more problems on the next page.

8. [12 pts] For each of the following determine where the statement is true (in all cases) or false (in at least one case) and prove yor claim. An answer without explanation will receive no credit.
   (a) Let R be a commutative domain and M an R-module. If $x, y \in M$ are torsion elements, then $x + y$ is also a torsion element.
   (b) If $\mathbb{F}$ and $\mathbb{K}$ are fields, then $\mathbb{F} \otimes_{\mathbb{Z}} \mathbb{K}$ is nonzero.
   (c) If R is a commutative ring and every R-module M is free, then R is a field.
   (d) If R is a commutative ring and M a finitely generated R-module, then every submodule of M is finitely generated.
9. [14 pts] Let $f = (x^3 - 2)(x^3 - 3) \in \mathbb{Q}[x]$. Let $\mathbb{K} \subseteq \mathbb{C}$ be the splitting field of f in the complex numbers.
   (a) Let $\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. Show that $\mathbb{Q}(\omega) \subseteq \mathbb{K}$.
   (b) Determine the Galois group $G = \mathrm{Gal}(\mathbb{K}/\mathbb{Q}(\omega))$. What common finite group is isomorphic to G? (E.g., "$\mathbb{Z}/7\mathbb{Z}$", "S_5", "A_{12}", etc.)