Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 6, 2024

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Recall that the DFT and inverse DFT are given by $\hat{y}_k = \sum_{j=0}^{n-1} y_j \bar{w}^{jk}$ and $y_j = \frac{1}{n}\sum_{j=0}^{n-1} \hat{y}_k w^{jk}$, where $w = e^{2\pi i/n}$.

(a) State and prove the Convolution Theorem for the DFT.

(b) Let a, x, y be column vectors with entries $a_0, \ldots, a_{n-1}$, $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1}$. In addition, let α, ξ and η be n-periodic sequences, the entries for one period, $k = 0, \ldots, n-1$, being those of a, x, and y, respectively. Consider the circulant matrix

$$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}.$$

Show that the matrix equation $Ax = y$ is equivalent to convolution $\eta = \alpha * \xi$.

(c) Use parts (a) and (b) above to show that the eigenvalues of A are the entries in $\widehat{\alpha}$.

Problem 2. Let $\mathcal{H}$ be a (separable) Hilbert space and and let S be a subset of $\mathcal{H}$.

(a) Define these: *S is compact subset of $\mathcal{H}$* and *S is a precompact subset of $\mathcal{H}$.*

(b) Consider the Hilbert space $\mathcal{H} = \ell^2$ and let

$$S = \{x = (x_1\ x_2\ x_3\ \ldots) \in \ell^2 : \sum_{n=1}^{\infty} n^2 |x_n|^2 < 1\}.$$

Show that S is a precompact subset of ℓ^2.

Problem 3. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$.

(a) State and prove the Fredholm alternative.

(b) State the Closed Range Theorem.

(c) Let $\mathcal{H}$ be $L^2[0,1]$ and consider the kernel $k(x,y) = x^4 y^{12}$ and its associated operator $Ku(x) = \int_0^1 k(x,y)u(y)dy$. Show that K is a Hilbert-Schmidt operator and that $\|K\|_{\text{op}} \le 1/15$.

(d) Consider the operator $L = I - \lambda K$. Find all values of λ such that $Lu = f$ has a unique solution for all $f \in L^2[0,1]$. For these values find the resolvent $(I - \lambda K)^{-1}$.

Problem 4. Let $\mathcal{D}$ and $\mathcal{D}'$ be the spaces of test functions and of distributions, respectively.

(a) Show that $\psi \in \mathcal{D}$ has the form $\psi = (x\phi)'$, where ϕ is also in $\mathcal{D}$, if and only if $\int_{-\infty}^{\infty} \psi(x)dx = \int_0^{\infty} \psi(x)dx = 0$.

(b) Find all $u \in \mathcal{D}'$ for which $\frac{d}{dx}\big(xu\big) = u$.

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APPLIED MATHEMATICS QUALIFIER: NUMERICAL ANALYSIS PART

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Problem 1. For this problem, you may use without proof Poincaré inequality and interpolation estimates as long as you accurately state them.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $f \in L_2(\Omega)$. Consider the function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega).$$

(1) State an additional assumption under which for every $f \in L_2(\Omega)$, we have $u \in H^2(\Omega)$ and $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L_2(\Omega)}$ for a constant C only depending on Ω. From now on we assume that such assumption holds.

(2) Consider a shape-regular and quasi-uniform sequence of triangulation $\{\mathcal{T}_h\}_{h>0}$ of Ω and design a conforming finite element approximation u_h of u such that

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\|f\|_{L_2(\Omega)},$$

where C only depends on Ω, the shape-regularity and quasi-uniformity constants. Justify your answer by proving the above estimate.

(3) Now let $z \in H_0^1(\Omega)$ be given by the relations

$$\int_\Omega \nabla z \cdot \nabla v = \int_\Omega v, \qquad \forall v \in H_0^1(\Omega).$$

Using a duality-type argument involving z show that

$$\int_\Omega (u - u_h) \leq Ch^2\|1\|_{L_2(\Omega)}\|f\|_{L_2(\Omega)},$$

where C only depends on Ω, the shape-regularity and quasi-uniformity constants. Be sure to clearly justify all the steps.

Problem 2. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $T > 0$, $f \in C^0(0,T;L_2(\Omega))$ and $u_0 \in L_2(\Omega)$. Consider the solution u to the parabolic problem

$$\frac{\partial}{\partial t}u - \Delta u = f \qquad \text{in } \Omega \times (0,T], \quad u = u_0 \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \times (0,T].$$

We assume that u is sufficiently smooth. We equip $H_0^1(\Omega)$ with the norm $\|v\|_{H_0^1(\Omega)} := \|\nabla v\|_{L_2(\Omega)}$ and let $H^{-1}(\Omega)$ be its dual space.

For $N \in \mathbb{N}$ and $\frac{1}{2} \leq \theta \leq 1$, consider the θ-method for the time approximation: Let $u^0 = u_0$ and for $n = 1, ..., N$, Define recursively $u^n \in H_0^1(\Omega)$ as the solution to

$$\frac{1}{\tau}\int_\Omega (u^n - u^{n-1})v + \int_\Omega (\theta\nabla u^n + (1-\theta)\nabla u^{n-1})\cdot\nabla v = \int_\Omega (\theta f(t_n) + (1-\theta)f(t_{n-1}))v, \qquad \forall v \in H_0^1(\Omega).$$

Here $\tau := T/N$ and $t_n := n\tau$.

Derive the following stability estimate for any $1 \leq m \leq N$

$$\|u^m\|_{L_2(\Omega)}^2 \leq \|u_0\|_{L_2(\Omega)}^2 + \tau\sum_{n=1}^{m}\|\theta f(t_n) + (1-\theta)f(t_{n-1})\|_{H^{-1}(\Omega)}^2$$

Hint: Recall that $(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2$ and $(a-b)b = \frac{1}{2}a^2 - \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2$.

Problem 3. For this problem, you may use without proof the Denis-Lions and Bramble-Hilbert lemmas as long as you accurately state them.

Let $K = [0,1]$, $\mathcal{P} = \mathbb{P}^2$ and $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$ where for $q \in \mathbb{P}^2$

$$\mathcal{N}_1(q) = q(0), \qquad \mathcal{N}_2(q) = q(1), \qquad \mathcal{N}_3(q) = \int_0^1 q.$$

(1) Prove or disprove that $(K, \mathcal{P}, \mathcal{N})$ is a finite element triplet.
(2) Find the dual basis of $\mathbb{P}^2$, i.e., $\{\lambda_1, \lambda_2, \lambda_3\}$ such that $\mathcal{N}_j(\lambda_i) = \delta_{ij}$, $1 \leq i, j \leq 3$.
(3) Define the finite interpolant $I_K : C^0(K) \to \mathcal{P}$ using the previously computed basis.
(4) Show that there is an absolute constant C such that for all $w \in H^3(K)$ there holds

$$\|w - I_K w\|_{L_2(K)} \leq C|w|_{H^3(K)}.$$

Problem 4. For $f \in C[0,1]$, we propose to approximate the solution to the following PDE

$$-u''(x) + u(x) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Let $N \in \mathbb{N}$, $h := 1/N$, $x_i := ih$ and $U_i \approx u(x_i)$ given by

$$-\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + U_i = f(x_i), \; i = 1, ..., N-1, \; U_0 = U_N = 0.$$

Show that

$$\max_{i=0,\ldots,N} |U_i| \leq \max_{i=1,\ldots,N} |f(x_i)|.$$

<u>*Hint:*</u> Argue for

$$U_k = \max_{i=1,\ldots,N-1} U_i \qquad \text{and} \qquad U_l = \min_{i=1,\ldots,N-1} U_i.$$