Complex Analysis
Qualifying Examination

August 2024

In this problem set, $\mathbb{D}$ is the open unit disk centered at zero.

1. Formulate the Little Picard Theorem; the Maximum Modulus Principle; Runge's Theorem. Sketch a proof of one of them.

2. The sequence z_n satisfies $|\operatorname{Im} z_n| < 1$, $\operatorname{Re} z_n \to +\infty$. Does this imply that $|\sin z_n|$ is bounded? Justify your answer.

3. Find a conformal map from $\{\operatorname{Re}(z) > 0\} \setminus \{|z-1| \leq 1\}$ onto $\mathbb{D}$.

4. Find the Laurent series expansion of the function
$$f(z) = \frac{1}{z(z-1)(z-2)}$$
valid in the region $\{1 < |z| < 2\}$.

5. The curve γ is the graph of cosine, given by $y = \cos x$ for $-\infty < x < \infty$ on the complex plane $z = x + iy$. Find:

   (a)
   $$\int_\gamma \frac{e^{iz}}{z^2+4} dz.$$

   (b)
   $$\int_\gamma \frac{e^{iz}}{(z^2+4)(e^{iz}-1)} dz$$

6. The function u is harmonic in $\mathbb{D}$, non-constant, and satisfies $u(0) = 0$. Show that for each $0 < r < 1$, the function u takes the value 0 on the circle $|z| = r$ at least twice.

7. The function f is analytic in the annulus $\{1 < |z| < 3\}$ and satisfies $f(z) = f(2z)$ for any $z \in \mathbb{C}$ such that both z and $2z$ are inside this annulus. Show that f is constant.

8. Polynomials p_n with $\deg p_n = n$ converge uniformly on compact subsets of $\mathbb{C}$ to a non-constant function f. Let $c_1^n, \ldots, c_n^n$ be roots of p_n, listed with multiplicity. Show that the sum of absolute values of all roots of p_n tends to infinity: $\lim_{n\to\infty} \sum_{k=1}^n |c_k^n| = \infty$.

9. Suppose that an analytic function $f\colon \mathbb{D} \setminus \{0\} \to \mathbb{C}$ has an essential singularity at zero and never takes the value 0. Which of the following is guaranteed to be true? Justify your answers.

   (a) $f(z) = e^{g(z)}$ for some analytic function g and any $z \in \mathbb{D} \setminus \{0\}$;

   (b) $f(e^z) = e^{g(z)}$ for some analytic function g and any z such that $\operatorname{Re} z < 0$.

10. Suppose that an entire non-constant, non-polynomial function f has the following property: at any point z with $f(z) = 0$, we have $f'(z) = 0$, $f''(z) \neq 0$.

    (a) Show that $f = g^2$ for an entire function g.

    (b) Using (a), show that f takes all complex values, except possibly zero, infinitely many times.

    Comment: You can get a full credit for (b) even if you did not solve (a).