TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
August 2023

- `There are 8 problems. Work on all of them and prove your assertions.`
- `Use a separate sheet for each problem and write only on one side of the paper.`
- `Write your name on the top right corner of each page.`

Problem 1 Let U be an open subset of a topological space. Is it true that U equals the interior of its closure? Justify your answer.

Problem 2 Let A be a proper subset of X and B be a proper subset of Y. If X and Y are connected, show that

$$X \times Y - A \times B$$

is connected. (Hint: Recall how "the product of two connected spaces is connected" is proved.)

Problem 3 Let X be a locally compact Hausdorff space. Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? Prove your answer.

Problem 4 Let $\mathbb{RP}^2$ be the real projective plane defined as the quotient space of the 2-dimensional sphere $\mathbb{S}^2$ by identifying the antipode points, i.e.,

$$\mathbb{RP}^2 = \mathbb{S}^2 / x \sim -x.$$

1. Compute the fundamental group of $\mathbb{RP}^2$.
2. Show that every continuous map from $\mathbb{RP}^2$ to the circle $\mathbb{S}^1$ is null-homotopic. [Hint: The lifting properties might be helpful here.]

Problem 5 Let Δ be the distribution on $\mathbb{R}^3 \setminus \{0\}$ so that, at the point $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$,

$$\Delta_{(x,y,z)} = \left\{ a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} : ax + by + cz = 0 \right\}.$$

Is Δ an involutive distribution? Why or why not?

Problem 6 Let σ be the 2-form

$$\sigma = \frac{x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

on $\mathbb{R}^3 \setminus \{0\}$.

1. Show that σ is closed, i.e., $d\sigma = 0$.
2. Let $i : \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ denote the inclusion map of the unit 2-sphere into $\mathbb{R}^3$. Find $\int_{\mathbb{S}^2} i^*\sigma$.

Problem 7 Let $f : \mathbb{R}^2 \to \mathbb{R}^4$ be given by

$$f(x,y) = (\cos x, \sin x, \cos y, \sin y)\,, \quad (x,y) \in \mathbb{R}^2.$$

1. Prove that f is an immersion.
2. The frame $e_1 = \frac{\partial f}{\partial x}$, $e_2 = \frac{\partial f}{\partial y}$ in $f(\mathbb{R}^2) \subset \mathbb{R}^4$ is orthonormal in the metric of $f(\mathbb{R}^2)$ induced by $\mathbb{R}^4$. Compute the dual coframe ω^1, ω^2 and the connection form ω^2_1.
3. Find the Gaussian curvature of the induced metric.

Problem 8 Suppose $\gamma : [0,\infty) \to M$ is an integral curve of a smooth vector field X on the smooth manifold M and suppose further that $\gamma(t)$ converges to a point $p \in M$ as $t \to \infty$. Prove that $X_p = 0$.