Algebra Qualifying Examination
12 August 2010

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. "Rough working" will not receive credit: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but may not be used as a step in your reasoning).
- All rings have an identity and all modules are unital.
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.

Notation: Throughout $\mathbb{Q}$ denotes the field of rationals and F and K denote fields. The multiplicative group of units in a ring R is denoted $R^\times$. The center of a group G is denoted $Z(G)$, and $N \lhd G$ means that N is a normal subgroup of G.

1. (10 points) Suppose G is a group with $|G| = 60$ and $|Z(G)|$ is divisible by 4. Show that G is abelian.

2. (10 points) Let M, N be modules over a ring R and $Hom(M, N)$ the set of R-module homomorphisms from M to N. For any $S \subset M$ define:

$$A(S) = \{\phi \in Hom(M, N) : \phi(S) = \{0\}\}.$$

   One of the following is always true for submodules M_1, M_2 of M:
   (i) $A(M_1 \cap M_2) = A(M_1) + A(M_2)$
   (ii) $A(M_1 + M_2) = A(M_1) \cap A(M_2)$
   Prove the true statement.

3. Let R be a ring with identity 1.
   (a) (8 points) Let M be a maximal ideal and $r \in R$ such that $1 - rx \in R^\times$ for all $x \in R$. Show that $r \in M$.
   (b) (8 points) (conversely) Suppose that $r \in M$ for every maximal ideal M. Show that $1 - rx \in R^\times$ for all $x \in R$. (Recall that every proper non-zero ideal is contained in some maximal ideal).

4. (10 points) Suppose that $f \in K[x]$ is irreducible of degree n and F is a field extension of K such that $[F : K] = m$ and $\gcd(m, n) = 1$. Show that f is irreducible over F.

5. Let $GL(n, F)$ denote the group of invertible $n \times n$ matrices with entries in F.
   (a) (8 points) Show that if $A \in GL(n, \mathbb{C})$ has finite order, then A is diagonalizable over $\mathbb{C}$ (that is, there is a basis for $\mathbb{C}^n$ with respect to which A is a diagonal matrix).
   (b) (8 points) Let p be a prime satisfying $p > n + 1$ (so p is odd). Show that if $A \in GL(n, \mathbb{Q})$ satisfies $A^p = I$ then $A = I$ (here I denotes the $n \times n$ identity matrix).

6. Find the degree of the splitting field of the polynomial $x^6 - 7$ over:
   (a) (6 points) $\mathbb{Q}$
   (b) (6 points) $\mathbb{Q}(\zeta_3)$ where ζ_3 is a primitive 3rd root of unity.
   (c) (6 points) $\mathbb{F}_3$ (the field with 3 elements).

7. (10 points) Let p be prime, $R = \mathbb{Z}_{p^n}$ the ring of integers modulo p^n, and A, B and C finitely-generated R-modules. Show that if $A \oplus B \cong A \oplus C$ then $B \cong C$.

8. (10 points) Let $p < q$ be primes and G a group of order pq^n. Show that G is solvable, that is, there exists subgroups N_i such that

$$G = N_0 \rhd N_1 \rhd \cdots \rhd N_r = (e)$$

such that N_{i-1}/N_i is abelian.