Algebra Qualifying Examination
11 August 2014

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. "Rough working" will not receive credit: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but may not be used as a step in your reasoning).
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.

Notation: Throughout, $\mathbb{Q}$ denotes the field of rational numbers, and $\overline{F}$ is the algebraic closure of the field F. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{F}_q$ denotes the field with q elements.

1. (12) Consider the matrix $M := \begin{pmatrix} 0 & 0 & -y \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ where y is an indeterminate.
   (a) Show that the characteristic polynomial $f_M(X)$ of M is irreducible in $\mathbb{Q}(y)[X]$.
   (b) Show that M is diagonalizable over the field $\overline{\mathbb{Q}(y)}$.
   (c) Show that M is not diagonalizable over the field $\overline{\mathbb{F}_3(y)}$.
2. (12) Suppose that $K \subset D \subset F$ where D is an integral domain and K, F are fields.
   (a) Show that if $[F : K] < \infty$ then D is a field.
   (b) Show, by example, that D can fail to be a field if $[F : K] = \infty$.
3. (12) Show that no group of order 90 is simple. (Hint: consider the normalizer of the intersection of two subgroups of order 9.)
4. (12) State and prove the Cayley-Hamilton Theorem for $T \in \mathrm{End}(V)$ with V a finite-dimensional vector space. (You many use any canonical form you wish.)
5. (16) Give an example of each of the following (no justification necessary)
   (a) A polynomial $f \in \mathbb{F}[x]$ with Galois group over $\mathbb{F}$ isomorphic to $\mathbb{Z}_5$ (you may choose $\mathbb{F}$).
   (b) A torsion-free $\mathbb{Z}$-module that is not free.
   (c) A projective module that is not free.

(d) A torsion injective $\mathbb{Z}$-module.

6. (12)
   (a) Let H be a finite p-group (p prime) acting on a finite set X with fixed points $X_0 \subset X$. Show that $|X| \equiv |X_0| \pmod{p}$.
   (b) Use part (a) to prove the second Sylow theorem: any two Sylow p-subgroups of a finite group G are conjugate. (Hint: let H and P be two Sylow p-subgroups, and let H act by translation on the cosets G/P.)
7. (12) Let F_1 and F_2 be finite dimensional Galois extension fields of K, such that $F_i \subset \overline{K}$ for some fixed algebraic closure $\overline{K}$ of K. Show that the compositum $F_1 \cdot F_2$ is also Galois over K.
8. (12) Let R be a commutative ring that satisfies the *descending chain condition* on ideals (*i.e.* for any chain of ideals $\cdots \subset I_k \subset \cdots \subset I_1$ there is a t such that $I_j = I_t$ for all $j \geq t$). Show that every prime ideal in R is maximal. (Hint: consider R/P for a prime ideal).