Texas A&M University Algebra Qualifying Exam

Tuesday, August 9, 2016

Instructions: There are 9 problems, some of which have multiple parts. Please attempt them all.
Also:

⊙ Please start each problem on a new page, clearly writing the problem number on that page.

⊙ Write your name on **each** page that you hand in. You are responsible for assuring that every page of your answers is handed in to the exam proctor.

⊙ Electronic or written assistance of any kind (such as calculators, cell phones, computers, notes, or books) is forbidden.

⊙ Please do not interpret any problem in a way that renders it trivial.

⊙ You must justify your answers fully. Merely stating a correct answer is not sufficient, nor is it enough to say that the problem is a known result of Abstract Algebra. When you do use a standard result, you must indicate which one. Your justifications must be clear and unambiguous. You will be graded not only on the correctness of your answers but on the quality of the proofs you provide.

⊙ Unless otherwise specified, we assume rings to be commutative, with multiplicative identity. Also, we assume all modules are unital left modules. By $\mathbb{N}$, we mean the nonnegative rational integers.

1. (**12 points**) Let G be a group of order 140. Prove that G has a cyclic normal subgroup of order 35.

2. (**10 points**) Let $f : R \to S$ be a homomorphism of commutative rings. Let P be a prime ideal of S and M a maximal ideal of S.

   (a) Prove that $f^{-1}(P)$ is a prime ideal of R.

   (b) If R is a subring of S, and f is the inclusion homomorphism, use (a) to prove that $P \cap R$ is a prime ideal of R.

   (c) Prove that, if f is surjective, then $f^{-1}(M)$ is a maximal ideal of R.

3. (**10 points**) Let R be a commutative ring. Let I be an ideal of R, and let J be the ideal of $R[x]$ generated by I.

   (a) Prove that $R[x]/J \cong (R/I)[x]$.

   (b) Prove that if I is a prime ideal of R, then J is a prime ideal of $R[x]$.

4. (**13 points**) Let $p_1, p_2, \ldots, p_n$, $n \geq 1$, be **distinct** primes in $\mathbb{N}$.

(a) Show that:

(i) For all $n \geq 1$, the field $K_n = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$ is Galois over $\mathbb{Q}$.
(ii) The Galois group $\mathrm{Aut}_{\mathbb{Q}}(K_n)$ of K_n over $\mathbb{Q}$ is isomorphic to

$$\underbrace{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \ldots \times \mathbb{Z}/2\mathbb{Z}}_{n \text{ copies}}.$$

(iii) There are $2^n - 1$ quadratic extensions of $\mathbb{Q}$ contained in K_n. Determine these fields explicitly in terms of the p_i, $i = 1, \ldots, n$.

(Hint: Prove (i),(ii) and (iii) together using induction. In order to get full credit you must give complete convincing proofs of (i), (ii), and (iii). Just saying "it follows by induction" is not sufficient.)

(b) Determine explicitly all quadratic extensions of $\mathbb{Q}$ contained in $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. You may use part (a).

5. (**10 points**) Suppose z is any generator for the unit group of $\mathbf{F}_{4^k}$, $k \geq 1$. Prove that $x^{2^k} + x + z^{2^k} + z$ has exactly 2^k roots in $\mathbf{F}_{4^k}$, $k \geq 1$. Here $\mathbf{F}_q$, $q \geq 2$, is the finite field with q elements.

6. (**12 points**)

(a) Let F_m be a free group of rank $m \geq 2$. Show that a nontrivial normal subgroup of F_m cannot be cyclic.

(b) Show that a solvable group cannot contain F_2 as a subgroup.

(Hint: A subgroup of a free group is free. You may also use part (a).)

7. (**10 points**) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

8. (**10 points**) Let $R = \mathbb{Z}[x]$, and consider the sequence of R-module homomorphisms

$$0 \to R \xrightarrow{f} R \xrightarrow{g} \mathbb{Z} \to 0$$

where, for $P = P(x) \in R$, we define $f(P) = xP(x)$ and $g(P) = P(0)$. Here $\mathbb{Z}$ is an R-module for the action of x on $1 \in \mathbb{Z}$ given by $x \cdot 1 = 0$.

(a) Show that the above sequence of R-modules is exact.

(b) Does it split as an exact sequence of R-modules?

(c) Does it split as an exact sequence of abelian groups?

9. (**13 points**) Let $f(x) = x^5 + x + 1 \in \mathbb{Q}[x]$.

(a) Find the degree $[K : \mathbb{Q}]$ of the splitting field K of $f(x)$ over $\mathbb{Q}$.

(b) Compute the Galois group $\mathrm{Aut}_{\mathbb{Q}}(K)$ of $f(x)$.

(You can use the fact that the discriminant of $x^3 - x^2 + 1$ equals 23.)

In both parts (a) and (b) simply writing down the correct answer is **not** sufficient: you must also justify your answer fully.