Texas A&M University Algebra Qualifying Exam
Wednesday, January 11, 2017

Instructions: There are 9 problems, please attempt them all.

- Please start each problem on a new page, clearly writing the problem number on that page.
- Write your name on **each** page that you hand in.
- You must justify your answers fully. Merely stating a correct answer is not sufficient, nor is it enough to say that the problem is a known result of Abstract Algebra. When you do use a standard result, you must indicate which one.
- Unless otherwise specified, we assume rings to be commutative, with multiplicative identity. Also, we assume all modules are unital left modules. By $\mathbb{N}$, we mean the nonnegative rational integers.

1. Prove that the quotient of S_4 by the Klein's group $\{e, (12)(34), (13)(24), (14)(23)\}$ is isomorphic to S_3.
2. How many Sylow 2-subgroups and Sylow 5-subgroups there are in a non-commutative group of order 20?
3. Consider the group $T = \{z \in \mathbb{C} \ : \ |z| = 1\}$ with respect to multiplication. Prove that every finite subgroup of T is cyclic.
4. Prove that every two-sided ideal of the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices is of the form $M_n(k\mathbb{Z})$ for some $k \in \mathbb{N}$.
5. Are the quotient rings $\mathbb{Z}[x]/(x^2 - 2)$ and $\mathbb{Z}[x]/(x^2 - 3)$ isomorphic?
6. Let R be a ring, and let M be a Noetherian left R-module. Suppose $\phi : M \to M$ is a surjective R-module homomorphism. Show that ϕ is an isomorphism. (Hint: Consider iterations ϕ, $\phi^2 = \phi \circ \phi$, $\phi^3 = \phi \circ \phi \circ \phi$, etc..)
7. Let R be an integral domain. Show that R is a field if and only if every R-module is projective. (Hint: for one direction, find an ideal I of R that is not prime and then consider R/I as an R-module.)
8. Let $\Bbbk$ be a field, $a \in \Bbbk$, and let p be a prime number. Prove that the polynomial $x^p + a$ is either irreducible or has a root in $\Bbbk$.
9. Let $g = (x^2 - 2)(x^2 + 3) \in \mathbb{Q}[x]$. Let E be the splitting field of g over $\mathbb{Q}$.
   (a) What is $[E : \mathbb{Q}]$?
   (b) Construct the Galois group $G = Gal(E/\mathbb{Q})$.
   (c) Show explicitly the correspondence between the intermediate fields $\mathbb{Q} \subseteq F \subseteq E$ and the subgroups $H \leq G$.