1. Let $H$ be a subgroup of a group $G$. Consider the normalizer and centralizer (respectively) of $H$:

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\} \quad \text{and} \quad C_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$ 

(a) (5 points) Prove that both the normalizer and centralizer are subgroups of $G$.
(b) (5 points) Prove that the centralizer is a normal subgroup of the normalizer.
(c) (5 points) Prove that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$ (the group of automorphisms of $H$, that is, bijective group homomorphisms from $H$ to itself).
(d) (5 points) Assume additionally that $H$ is a normal subgroup of $G$, and that $H$ is finite. Prove that the index of $C_G(H)$ in $G$ is finite.

2. (10 points) Recall that by definition, a commutative ring $R$ is local if $R$ has a unique maximal ideal. Prove that a commutative ring $R$ is local if and only if for all $r, r' \in R$, if $r + r' = 1_R$ then $r$ or $r'$ is a unit.

3. (10 points) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of $R$-modules. Let $\text{id}_A, \text{id}_C$ denote the identity maps on $A, C$, respectively. Consider the following statements:

(i) There is an $R$-module homomorphism $\phi : C \to B$ such that $\beta \circ \phi = \text{id}_C$.
(ii) There is an $R$-module homomorphism $\psi : B \to A$ such that $\psi \circ \alpha = \text{id}_A$.
Prove that (i) implies (ii). (Note it is also true that (ii) implies (i).)
4. Let $R$ be a commutative ring and let $M$ be an $R$-module. Let $T(M)$ be the set of torsion elements of $M$, that is, $T(M) = \{ m \in M \mid r \cdot m = 0 \text{ for some nonzero } r \in R \}$.

(a) (5 points) Prove that if $R$ is an integral domain, then $T(M)$ is an $R$-submodule of $M$.

(b) (5 points) Give an example of a ring $R$ and an $R$-module $M$ for which $T(M)$ is not an $R$-submodule of $M$.

(c) (5 points) Let $M, N$ be $R$-modules, and let $f : M \to N$ be an $R$-module homomorphism. Prove that $f(T(M)) \subseteq T(N)$.

5. (10 points) Let $R$ be a commutative ring and let $I, J$ be ideals of $R$. Prove that there is an $R$-module isomorphism $(R/I) \otimes_R (R/J) \cong R/(I + J)$.

6. The goal of this problem is to prove that $\mathbb{C}$ is an algebraically closed field. (So, do NOT use this fact in your solution!)

   (a) (5 points) Let $K/\mathbb{R}$ be a finite extension. Prove that if $[K : \mathbb{R}]$ is odd, then $K = \mathbb{R}$.

   (b) (5 points) Let $L/\mathbb{R}$ be a finite Galois extension of $\mathbb{R}$. Prove that $[L : \mathbb{R}]$ is a power of 2. (Hint: Sylow’s Theorem)

   (c) (5 points) Prove that there is no extension $K/\mathbb{C}$ with $[K : \mathbb{C}] = 2$.

   (d) (5 points) Let $K/\mathbb{C}$ be any finite extension. Show that there is some finite Galois extension $L/\mathbb{R}$ with $\mathbb{R} \subseteq \mathbb{C} \subseteq K \subseteq L$. Show that $L = \mathbb{C}$, and deduce that $K = \mathbb{C}$.

7. Let $F$ be a finite field, let $f$ be a monic irreducible polynomial in $F[x]$, and let $\alpha \in F$ be a root of $f$. Prove the following:

   (a) (5 points) $F(\alpha)$ is the splitting field for $f$ over $F$, and

   (b) (5 points) the set of roots of $f$ is $\{ \alpha^{F|r} \mid r \geq 1 \}$.

8. (10 points) Is the symmetric group $S_4$ the internal direct sum of two or more nontrivial subgroups? Prove your answer.