Algebra Qualifying Examination
January 14, 2021

Instructions:

• There are nine problems worth a total of 100 points. Individual point values are listed by each problem.

• Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. Answers must be written in a structured and understandable manner.

• Attach the test to your work and write your name on the top of the first page of the test. Do not forget to staple your work to the test at the top using a stapler.

Notation: Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers.

1. (8)
   (a) Prove that an extension of a torsion group by a torsion group is torsion. I.e. if $G/N = H$ and H, N are torsion groups then G is torsion.
   (b) Prove that an extension of a torsion–free group by a torsion–free group is torsion–free.

2. (10) Let G be a group and G' be the commutator subgroup i.e. the subgroup generated by elements $[a, b] = aba^{-1}b^{-1}, a, b \in G$.
   (a) Prove that G' is a normal subgroup in G.
   (b) Prove that $G_{ab} = G/G'$ is abelian.
   (c) Prove that if $N \lhd G$ and G/N is abelian then N contains G'.

3. (11) Classify all groups of order 21.

4. (11) Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a map given by $F(x, y) = (2x + y, x + y), x, y \in \mathbb{C}$. Consider $\mathbb{C}^2$ to be a $\mathbb{C}[t]$-module by letting $p(t) \cdot v = p(F)(v)$ for any $p(t) \in \mathbb{C}[t], v \in \mathbb{C}^2$.
   (a) Is $\mathbb{C}^2$ a cyclic $\mathbb{C}[t]$-module?
   (b) Find all the $\mathbb{C}[t]$-submodules of $\mathbb{C}^2$.

5. (12) Let $R = \mathbb{Z}[\sqrt{-5}]$ be the ring of complex numbers of the form $a+bi\sqrt{5}, a, b \in \mathbb{Z}, i = \sqrt{-1}$. Which of the following statements are true for R?
   (a) R is an integral domain.
   (b) R is a Euclidian domain
   (c) R is a unique factorization domain.
   Justify your answers.

6. (12) A group G is divisible if for any $g \in G$ and positive integer n there is an $x \in G$ such that $x^n = g$ (if G is abelian group then in additive notations this means that $nx = g$).
   (a) Give one example of divisible abelian groups.
   (b) Prove that the direct product of divisible groups is a divisible group.
   (c) Give the definition of injective R-module where R is a ring.
   (d) Prove that every injective $\mathbb{Z}$-module is a divisible abelian group.

7. (10) Prove that every nonzero nonunit in a principal ideal domain is the product of finitely many irreducible elements.

8. (12) Consider the matrix $M = \begin{bmatrix} 0 & 0 & y \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (y is an indeterminate).
   (a) Is the characteristic polynomial of M irreducible in $\mathbb{Q}(y)[x]$?
   (b) Is M diagonalizable over the field $\overline{\mathbb{Q}(y)}$? ($\overline{\mathbb{Q}(y)}$ denotes an algebraic closure of $\mathbb{Q}(y)$).

9. (14) Consider the polynomial $f(x) = x^5 - 4x - 2$ over $\mathbb{Q}$ and its splitting field F.
   (a) Prove that f is irreducible over $\mathbb{Q}$.
   (b) Determine the number of real roots of f.
   (c) Prove that the Galois group G of the extension F over $\mathbb{Q}$ is a subgroup of the symmetric group S_5.
   (d) Prove that G contains a transposition.
   (e) Prove that G contains a 5-cycle.
   (f) Determine G.
   (g) Is the equation $x^5 - 4x - 2 = 0$ solvable by radicals over $\mathbb{Q}$?