Algebra Qualifying Examination
January 2022

Instructions:

- There are 8 problems worth a total of 100 points. Individual point values are listed by each problem.
- Credit awarded for your answers will be based upon the correctness of your answers as well as the clarity and main steps of your reasoning. "Rough working" will not receive credit: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but may not be used as a step in your reasoning).
- Every effort is made to ensure that there are no typographical errors or omissions. If you suspect there is an error, check with the exam administrator. Do not interpret the problem in a way that makes it trivial.

Notation: Throughout, $\mathbb{Q}$ denotes the field of rational numbers, and $\overline{F}$ is the algebraic closure of the field F. $\mathbb{Z}$ denotes the ring of integers and $M_n(R)$ denotes the ring of matrices with entries in the ring R.

1. (12) Prove or disprove: Let F be a field. If $A \in M_n(F)$ has finite order ($A^k = I$) then A is diagonalizable over some extension field $K \supset F$.
2. (12) Show that $R = \mathbb{Q}[X, Y]/\langle X^9 + Y^4 - 1\rangle$ is an integral domain.
3. (12) Denote by x, y the images of X, Y under the homomorphism $\mathbb{Q}[X, Y] \to R$ in 2. (You may use the result of Problem 2. whether you could prove it or not.)
   (a) Show that the composition $\mathbb{Q}[X] \to \mathbb{Q}[X, Y] \to R$ is injective (so that $\mathbb{Q}[X] \cong \mathbb{Q}[x]$).
   (b) Show that R is a free $\mathbb{Q}[X]$-module.
4. (12) Show that there are (at least) 3 non-isomorphic non-abelian groups of order $52 = 4 \cdot 13$.
5. (12) Give an example of each of the following (no justification necessary)
   (a) A local ring with characteristic $p > 0$.
   (b) A non-solvable group.
   (c) A finite dimensional non-Galois extension of $\mathbb{Q}$.
   (d) An integral domain that is not a UFD.
6. (12) Consider the $\mathbb{Z}[x]$-module A generated by v_1, v_2 with action defined by $x.v_1 = -3v_1 - 2v_2$ and $x.v_2 = 2v_1 + 2v_2$. Show that A is not the direct sum of two submodules (hint: it has a non-trivial submodule).

7. (16) Let L be the splitting field of x^6+2 over $\mathbb{Q}$.
   (a) What is $[L:\mathbb{Q}]$?
   (b) Show that L contains the field $\mathbb{Q}(\sqrt{-3})$.
   (c) L contains three distinct subfields K_1, K_2, K_3 which are quadratic over $\mathbb{Q}$ (i.e., $[K_i:\mathbb{Q}]=2$ for each i). What are they?
   (d) What is the fixed field of L by complex conjugation?
8. (12) Let R be a commutative ring and $f \in R$ a non-nilpotent element. Consider the set of ideals $\mathcal{A} = \{I < R : f^k \notin I \text{ for all } k \in \mathbb{N}\}$. Prove that any maximal element of $\mathcal{A}$ (assuming one exists) is a prime ideal in R.