Algebra Qualifying Examination
January 2009

Instructions:

- There are nine Algebra questions worth a total of 100 points. Individual point values are listed next to each problem.
- Credit awarded for your answers will be based on the correctness of your answers as well as the clarity and main steps of your reasoning. "Rough working" will not be accepted: Answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).

Notation: Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers.

1. (10 points) Prove that a group of order 182 is solvable. (Note that $182 = 2 \cdot 7 \cdot 13$.)

2. (10 points) Let G be a finite group and N a normal subgroup of G. Let $\mathcal{C}$ be a conjugacy class of G that is contained in N. Prove that if $|G : N| = p$ is prime, then either $\mathcal{C}$ is a conjugacy class of N or $\mathcal{C}$ is a union of p distinct conjugacy classes of N.

3. (10 points) Let $i = \sqrt{-1}$ in $\mathbb{C}$, and let x be an indeterminate.
   (a) Show that the three *additive groups* $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Z}[x]/(x^2)$ are all isomorphic to each other.
   (b) Show that no two of the three *rings* $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Z}[x]/(x^2)$ are isomorphic to each other.

4. (10 points) Let R be a commutative ring with 1. Show that the sum of any two principal ideals of R is principal if, and only if, every finitely generated ideal of R is principal.

5. (10 points) Let R be the subring of the polynomial ring $\mathbb{Z}[x]$ consisting of every polynomial in which the coefficients of x and x^2 are both 0. Prove that the field of fractions of R is precisely the field of rational functions $\mathbb{Q}(x)$ over $\mathbb{Q}$.

6. (10 points) Let V be a finite dimensional vector space over an algebraically closed field F, and let S and T be two linear transformations from V to V. Assume that $ST = TS$ and that the characteristic polynomial of S has distinct roots.
   (a) Show that every eigenvector of S is an eigenvector of T.
   (b) If T is nilpotent, show that $T = 0$.

7. (10 points) Let R be a ring with 1. Consider R to be a left R-module via multiplication. Prove that R is a division ring if, and only if, R is simple as an R-module.

8. (15 points)
   (1) Let F be a field. Let α and β be algebraic over F with minimal polynomials f and g respectively. Show that g is irreducible over $F(\alpha)$ if and only if f is irreducible over $F(\beta)$.
   (2) Let L be the splitting field of $x^4 - 6$ over $\mathbb{Q}$. Determine $[L : \mathbb{Q}]$.

9. (15 points) Let F be a field of characteristic not 2, and let $a, b \in F$. Let L be the splitting field of $(x^2 - a)(x^2 - b)$.
   (1) Suppose that none of a, b, or ab are perfect squares in F. Show that $\mathrm{Gal(L/F)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
   (2) Is the converse of part (a) true? Prove or disprove.