Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 11, 2016

Instructions: Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.

Problem 1. Let $\mathcal{D}$ be the set of compactly supported $C^\infty$ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = \phi''(x)$ for some $\phi \in \mathcal{D}$ if and only if $\int_{-\infty}^{\infty} \psi(x) dx = \int_{-\infty}^{\infty} x\psi(x) dx = 0$.
(d) Use 2(c) to solve, in the distributional sense, the differential equation $u'' = 0$.

Problem 2. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having $u''$ in $L^2[0,\infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, $L$ has the domain $\mathcal{D}_L = \{u \in L^2[0,\infty) \mid u'' \in L^2[0,\infty) \text{ and } u'(0) = 0\}$.

(a) Find the Green's function $G(x,\xi; z)$ for $-G'' - zG = \delta(x - \xi)$, with $G_x(0,\xi; z) = 0$.
(b) Employ the spectral theorem (Stone's formula) to obtain the cosine transform formulas:

$$F(\mu) = \frac{2}{\pi} \int_0^\infty f(x) \cos(\mu x) dx \quad \text{and} \quad f(x) = \int_0^\infty F(\mu) \cos(\mu x) d\mu.$$ 

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) Consider $K \in \mathcal{C}(\mathcal{H})$. Show that if $\{\phi_n\}_{n=0}^\infty$ is an orthonormal set in $\mathcal{H}$, then $\lim_{n \to \infty} K \phi_n = 0$.
(b) Suppose that $K \in \mathcal{C}(\mathcal{H})$ is self adjoint.

(i) Show that $\sigma(K)$ (the spectrum) consists only of eigenvalues, together with 0, and that the only limit point of $\sigma(K)$ is 0.
(ii) Given that $\|K\| = \sup_{\|u\|=1} \langle Ku, u \rangle$, show that either $\|K\|$ or $-\|K\|$ (or possibly both) is an eigenvalue of $K$, and that the corresponding eigenspace is finite dimensional.

Problem 4. Suppose that $f(x)$ is 2$\pi$-periodic function in $C^{m}(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2$\pi$-periodic. Here $m > 0$ is a fixed integer. Let $c_k$ denote the $k^{th}$ (complex) Fourier coefficient for $f$, and let $c_k^{(j)}$ denote the $k^{th}$ (complex) Fourier coefficient for $f^{(j)}$.

(a) Prove that $c_k^{(j)} = (ik)^j c_k$, $j = 0,\ldots,m+1$. (Note: using term by term differentiation of the Fourier series assumes what you want to prove.)
(b) For $k \neq 0$, show that $c_k$ satisfies the bound

$$|c_k| \leq \frac{1}{2\pi |k|^{m+1}} \|f^{(m+1)}\|_{L^1[0,2\pi]}.$$ 

(c) Let $f_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ be the $n^{th}$ partial sum of the Fourier series for $f$, $n \geq 1$. Show that

$$\|f - f_n\|_{L^2[0,2\pi]} \leq C \frac{\|f^{(m+1)}\|_{L^1[0,2\pi]}}{n^{m+\frac{1}{2}}} ,$$

where $C$ is independent of $f$ and $n$. 

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APPLIED MATHEMATICS/NUMERICAL ANALYSIS QUALIFIER

Aug. 11, 2016

Numerical Analysis part, 2 hours

Problem 1. Let $K \subset \mathbb{R}^2$ be a simplex. Let $k \in \mathbb{N}$ and consider the set of multi-indices $\mathcal{A}_{k,2} := \{ \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid |\alpha| \leq k \}$. Let $\mathbb{P}_{k,2}$ be the set of the real-valued 2-variate polynomials of degree at most $k$.

1. Let $\Sigma_{k,2}$ be the collection of the following linear forms $\sigma_\alpha(p) = \int_K \partial^{\alpha_1} \partial^{\alpha_2} p \, dx$, for all $\alpha \in \mathcal{A}_{k,2}$ and all $p \in \mathbb{P}_{k,2}$, where $\partial^{\alpha_i} p(x_1, x_2)$ is the $\alpha_i$-th partial derivative of $p$ with respect to $x_i$. Show that $(K, \mathbb{P}_{k,2}, \Sigma_{k,2})$ is a finite element. (Hint: induction on $k$.)

2. Let $f \in W^{k,1}(K)$. Show that there exists a unique $q \in \mathbb{P}_{k,2}$ such that $\int_K \partial^{\alpha_1} \partial^{\alpha_2} (q - f) \, dx = 0$ for all $\alpha \in \mathcal{A}_{k,2}$.

3. Compute the three shape functions of $(\tilde{K}, \mathbb{P}_{1,2}, \Sigma_{1,2})$ where $\tilde{K} := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1, 0 \leq x_2, x_1 + x_2 \leq 1\}$.

Problem 2. Consider the equation $\mu \partial_t u + \beta \partial_x u - \nu \partial_{xx} u = f$ in $D = (0, 1)$, $t > 0$, where $\mu \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, $\nu \in \mathbb{R}_+$, and $f \in L^2(D)$ with boundary conditions $u(0) = 0$, $u(1) = 0$ and initial data $u(x, 0) = 0$. Let $\mathcal{T}_h$ be the mesh composed of the cells $[ih, (i+1)h]$, $i \in \{0 : I\}$, with uniform mesh size $h = \frac{1}{I+1}$. Let $P(\mathcal{T}_h)$ be the finite element space composed of continuous piecewise linear functions that are zero at 0 and at 1. Let $(\varphi_i)_{0 \leq i \leq I}$ be the global Lagrange shape functions associated with the nodes $x_i = ih$, $i \in \{1 : I\}$.

1. Write the fully discretize version of the problem in $P(\mathcal{T}_h)$ using the implicit Euler approximation for the time. Denote the time step by $\Delta t$ and $t^l := l \Delta t$ for all $l \in \mathbb{N}$.

2. Prove one stability estimate. (Hint: You may want to introduce the Poincaré constant $c_P$ such that $c_P \|v\|_{L^2} \leq \|\partial_x v\|_{L^2}$ for all $v \in H^1_0(D)$.)

3. Denoting by $u^l_k = \sum_{0 \leq i \leq I+1} U^l_i \varphi_i$ the approximation of $u$ at time $t^l = l \Delta t$, write the linear system solved by $(U_1^{n+1}, \ldots, U_I^{n+1})^T$.

Problem 3. Let $D = (0, 1)$ and $f(x) = \frac{1}{x(1-x)}$. Consider the problem:

$-\partial_x ((1 + \sin(x)^2) \partial_x u) = f$ in $D$ with $u(0) = u(1) = 0$.

1. Prove that $f$ is the weak derivative of $g(x) = \log(x) - \log(1-x)$.

2. Is $g$ in $L^2(D)$?

3. Write a weak formulation of the above problem with both trial and test spaces equal to $H^1_0(D)$.

4. Show that the problem is well posed in $H^1_0(D)$.

5. Let $\mathcal{T}_h$ be the mesh composed of the cells $[ih, (i+1)h]$, $i \in \{0 : I\}$, with uniform mesh size $h = 1/I$. Let $P(\mathcal{T}_h)$ be the finite element
space composed of continuous piecewise linear functions that are zero at 0 and 1. Write the discrete problem in \( P(T_h) \).

(5) Derive an error estimate in \( H^1(D) \). (Note that we only have \( u \in H^{r_{\text{max}}}(D) \) for some \( r_{\text{max}} \in (1,2) \) since \( f \) is not in \( L^2(D) \)).

(6) Derive an improved error estimate in \( L^2(D) \). (Hint: Use a duality argument. Consider the problem \(-\Delta u((1 + \sin(x)^2)\partial_u s) = v\) in \( D \) with \( s(0) = s(1) = 0 \) and \( v \in L^2(D) \). Accept as a fact that \( s \) is in \( H^2(D) \) if \( v \in L^2(D) \), and there is \( c > 0 \) such that \( |s|_{H^2} \leq c||v||_{L^2} \) for all \( v \in L^2(D) \).)

(7) Bonus question if you have time. Prove the elliptic regularity statement in the above hint.