Complex analysis qualifying exam, August 2010.

1. Give the statements of the following:

   (a) Montel's theorem;
   (b) Schwarz's lemma.

2. Let f be a meromorphic function in a neighborhood of the closed unit disk $\bar{\mathbb{D}} = \{|z| \leq 1\}$. Suppose that $\Im f$ does not have zeros on the unit circle $\mathbb{T} = \{|z| = 1\}$. Prove that then the number of zeros of f inside $\mathbb{T}$ is equal to the number of poles of f inside $\mathbb{T}$.

3. Calculate "the Fresnel integrals,"

$$\int_0^\infty \sin(x^2)dx \quad \text{and} \quad \int_0^\infty \cos(x^2)dx,$$

that play an important role in diffraction theory. (You can use the value for the Gaussian integral: $\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}dx = \sqrt{2\pi}$.)

4. Let f be a bounded holomorphic function in the unit disk $\mathbb{D} = \{|z| < 1\}$. Suppose that the radial limits of f are zero on a nontrivial arc of the unit circle, i.e. that for some $0 \leq \alpha < \beta \leq 2\pi$,

$$\lim_{r \to 1-} f(re^{i\gamma}) = 0$$

for all $\alpha < \gamma < \beta$. Prove that then f is identically zero. (This is a simple version of a theorem proved by F. and M. Riesz in 1916.)

5. Suppose that the point w in the complex plane $\mathbb{C}$ is moving according to the law $w = re^{it}$, where r is a constant and t is the time variable. Let f be a function, holomorphic in a neighborhood of the circle $\{|z| = r\}$. Find the instantaneous velocity of the point $f(w)$ at the moment t. (The answer should be presented in a form of a two-dimensional vector, depending on t and f.)

6. For an entire function f denote

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

Prove that for any $0 < \alpha < 1$, the limit

$$\lim_{r \to \infty} \frac{M_f(\alpha r)}{M_f(r)}$$

is equal to α^n if f is a polynomial of degree n and to 0 if f is transcendental (not a polynomial).

7. Let f be a meromorphic function in a neighborhood of the closed unit disk $\bar{\mathbb{D}}$. Suppose that f is holomorphic in $\mathbb{D}$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (*)$$

for $z \in \mathbb{D}$. Prove that if f has a pole on the unit circle $\mathbb{T}$ then the series in $(*)$ diverges at any $z \in \mathbb{T}$.

8. Find a general formula for all functions $w(z)$ that map the domain $\{|\Im z| < 1,\ \Re z > 0\}$ conformally onto the domain $\mathbb{D} \setminus [0, 1]$.

9. Let $u_n,\ n = 1, 2, ...$ be a sequence of harmonic functions in a complex domain Ω such that $|u_n(z)| < C$ in Ω for some $C > 0$ and all n. Suppose that there exists a subdomain $\Gamma \subset \Omega$ such that the sequence $u_n(z)$ converges for any $z \in \Gamma$. Prove that then $u_n(z)$ converges for any $z \in \Omega$ and the limit function is harmonic in Ω.

10. Let f be meromorphic in a complex domain Ω. Prove that then f' is also meromorphic in Ω.