Complex Analysis Qualifying Examination

August 2013

1. Let $u : \mathbb{C} \to \mathbb{R}$ be a continuous function such that e^u is a harmonic function. Prove that u must be a constant function.

2. Suppose that f is holomorphic in $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$, the punctured unit disk. Prove that the point 0 is a removable singularity for the function f if and only if the point 0 is a removable singularity for the function $f' f''$ (the product of the first derivative of f and the second derivative of f).

3. Show that
$$\int_0^{2\pi} \frac{\cos(\theta)}{1 - \cos(\theta) + \frac{1}{4}} \, \mathrm{d}\theta = \frac{4\pi}{3}.$$

4. Suppose that $p(z)$ and $q(z)$ are polynomials of degree 2013, and all the zeroes of p lie inside the open unit disk. Prove that if $|q(z)| \leq |p(z)|$ whenever $|z| = 1$, then $|q(z)| \leq |p(z)|$ whenever $|z| > 1$.

5. Does there exist a sequence $\{p_n(z)\}_{n=1}^{\infty}$ of polynomials such that $\lim_{n\to\infty} p_n(z) = 0$ when $z = 0$ and $\lim_{n\to\infty} p_n(z) = 1$ when $z \neq 0$? Explain why or why not.

6. Suppose f is an entire function that is odd [antisymmetric: that is, $f(z) = -f(-z)$ for every z]. Show that $f(\mathbb{C})$, the range of f, is either $\mathbb{C}$ or $\{0\}$.

7. Suppose $\Omega = \mathbb{C} \setminus \{ z : \operatorname{Re}(z) = 0 \text{ and } -1 \leq \operatorname{Im}(z) \leq 1 \}$ (the complex plane with a slit along the imaginary axis from $-i$ to i). There exists a holomorphic function f on Ω with the property that $\big(f(z)\big)^2 = z^2 + 1$ for every z in Ω. Prove that f is odd: namely, $f(z) = -f(-z)$ for every z in Ω.

8. Show that the function sending z to $i \tan(iz)$ is a biholomorphic mapping from the horizontal strip $\{ z \in \mathbb{C} : -\pi/2 < \operatorname{Im}(z) < \pi/2 \}$ onto the complex plane with slits along the real axis from $-\infty$ to -1 and from 1 to ∞.

9. Does there exist an entire function that takes every complex value on every line through the origin? Explain why or why not.

10. State *one* of the following theorems and sketch the proof: the Riemann mapping theorem, Mittag-Leffler's theorem, or Morera's theorem.