<u>Complex Analysis Qualifying Exam, August 2020</u>

<u>Problem 1:</u> Let f be analytic in $\{z \in \mathbb{C} \,|\, |z| < 1\}$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show that $|f'(0)| \leq 1 - |f(0)|^2$.

<u>Problem 2:</u> Assume a sequence $\{f_n\}_{n=1}^{\infty}$ of meromorphic functions in a region $\Omega \subseteq \mathbb{C}$ converges in $C(\Omega, \mathbb{C}_\infty)$ to an analytic function f. Show that for every compact subset K of Ω, there is an integer n_0 such that the poles of f_n lie in $\Omega \setminus K$ for $n \geq n_0$.

<u>Problem 3:</u> Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ when $|z| < 1$. Show that if $|a_4| = 2^4 \max_{|z|=1/2} |f(z)|$, then $f(z) = a_4 z^4$.

<u>Problem 4:</u> Use the argument principle to prove that a nonconstant polynomial of degree n has exactly n zeros when counted with multiplicity.

<u>Problem 5:</u> Show that

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx = \pi \ .$$

<u>Problem 6:</u>

a) State the Monodromy Theorem, the Mittag–Leffler Theorem, and Montel's normality criterion.

b) Sketch a proof of *one* of the theorems in a).

<u>Problem 7:</u> Denote by Ω the union of the annuli $\{z \in \mathbb{C} \,|\, 1 < |z| < 2\}$ and $\{z \in \mathbb{C} \,|\, 1 < |z - 3| < 2\}$, with the two intervals $(-2, -1)$ and $(4, 5)$ removed.

a) Show that Ω is simply connected.

b) In view of a), every analytic function in Ω can be approximated locally uniformly by polynomials (Why?). If the analytic function is bounded, can the approximating sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials be chosen so that $\sup_{n\in\mathbb{N}} \sup_{z\in\Omega} |p_n(z)| < \infty$? Justify your answer.

<u>Problem 8:</u> Let Ω be an open subset of $\mathbb{C}$, $\Omega \neq \mathbb{C}$, and suppose $\{c_j\}_{j=1}^{\infty}$ is dense in the boundary of Ω. Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $c_1, c_1, c_2, c_1, c_2, c_3, \cdots$, so that each c_j is repeated infinitely often. Show that if $\{b_n\}_{n=1}^{\infty} \subset \Omega$ is a

sequence such that $\sum_{n=1}^{\infty} |a_n - b_n| < \infty$, then $\Pi_{n=1}^{\infty} \frac{z-b_n}{z-a_n}$ represents an analytic function in Ω which cannot be continued analytically past any boundary point.

Problem 9: Prove that the equation $e^z - z^{2020} - 1 = 0$ has infinitely many solutions.

Problem 10: Suppose a and b are distinct points in the complex plane. Let A_1 be the family of all circles and lines through the pair of points a and b, and let A_2 be the family of all circles and lines with respect to which a and b are symmetric. Show that if c is a third point in the plane, then there is precisely one element of A_1 passing through c and precisely one element of A_2 passing through c, and these two curves meet orthogonally.