Complex Analysis Qualifying Exam, August 2021

Problem 1: Denote by L the real axis, by C, C_1, and C_2 the circles $\{|z+i|=1\}$, $\{|z+1|=1\}$, and $\{|z-1|=1\}$, respectively.

Find a Möbius transformation that takes the pair (L, C) to the pair (C_1, C_2). (Explain why the function you found has the required properties.)

Problem 2: Assume f is analytic on the unit disc and $\lim_{r\to 1^-} f(re^{i\theta}) = 0$, uniformly for $0 < \theta < \pi/3$. Show that $f \equiv 0$.

Problem 3: Suppose G is a bounded region. Let L be a straight line, so that $G \cap L$ is an interval on L. L divides $\mathbb{C}$ into two open half planes Π_1 and Π_2. Set $G_1 = \Pi_1 \cap G$ and $G_2 = \Pi_2 \cap G$. Show: if G_1 and G_2 are simply connected, then so is G.

Problem 4: Suppose f and g are analytic in a neighborhood of the closed unit disc $\overline{\mathbb{D}}$. Suppose further that f has a simple zero at $z_0 = 0$, and no other zeros on $\overline{\mathbb{D}}$. Set $f_\varepsilon(z) := f(z) + \varepsilon g(z)$. Show that if $|\varepsilon|$ is sufficiently small, then

a) f_ε has a unique zero in $\mathbb{D}$, and

b) if z_ε is this zero, then the map $\varepsilon \to z_\varepsilon$ is continuous.

Problem 5: Show that for $\xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}\, dx = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|} \quad .$$

Problem 6: Let G be a bounded open set and $f : G \to G$ an analytic function. Complete the outline below to prove that if f has a fixed point z_0 (i.e. $f(z_0) = z_0$) with $f'(z_0) = 1$, then $f(z) \equiv z$.

a) You may assume that $z_0 = 0$.

b) Then, near 0, $f(z) = z + a_n z^n + \cdots$, $n \geq 2$. Show that $\underbrace{f \circ f \circ f \cdots \circ f(z)}_{k \text{ times}} = z + k a_n z^n + \cdots$.

c) There is a constant C so that $k|a_n| \leq C$, $\forall k$.

Problem 7: Show that the series $\sum_{n\geq 1} \frac{(-1)^n}{z+n}$ defines an analytic function in $\mathbb{C} \setminus \{-1, -2, -3, \cdots\}$.

Problem 8: Let G be a domain, $f : G \to G$ biholomorphic, $z_0 \in G$, and γ a closed rectifiable curve homologous to zero in G. For $z_0 \in G \setminus \gamma$, show that the index $n(\gamma; z_0)$ equals $n(f(\gamma); f(z_0))$.

Problem 9: Let f be a meromorphic function on $\mathbb{C}$. Assume there are constants M, r, and an integer n such that $|f(z)| \leq M|z|^n$ for $|z| > r$, z not a pole. Show that f is a rational function.

Problem 10: State Mittag–Leffler's theorem, Runge's theorem, and the Riemann mapping theorem.