Complex analysis qualifying exam, January 2018.

1. Give the statements of the following theorems:
   (a) Montel's theorem;
   (b) Runge's theorem.

2. Let $f$ be holomorphic in the unit disk $D$ and continuous in the closure of $D$. Prove that
   \[
   \int_0^1 f(x)dx = \frac{1}{2\pi i} \int_{|z|=1} f(z) \log zdz,
   \]
   where $\log$ denotes the branch of logarithm whose imaginary part takes values in $(0, 2\pi)$.

3. Suppose that there exists a conformal map from $\{0 \leq r_1 < |z| < r_2\}$ to $\{0 \leq R_1 < |z| < R_2\}$. Prove that $r_1/r_2 = R_1/R_2$.

4. Does there exist a bounded holomorphic function in the unit disk $D$ such that
   \[
   f \left( 1 - \frac{1}{n} \right) = \frac{(-1)^n}{n}
   \]
   for $n = 1, 2, 3, \ldots$?

5. Calculate the integral
   \[
   \int_{|z|=5} \frac{z}{e^z - i} dz.
   \]

6. Find the disk of convergence for the power series
   \[
   f(z) = 1 + z^2 + z^{2^2} + \ldots + z^{2^n} + \ldots
   \]
   Show that $f(z)$ cannot be analytically continued to any connected domain properly containing the disk of convergence.

7. Show that the range of the entire function $\frac{\sin z}{z}$ is the whole complex plane $\mathbb{C}$.

8. Find a conformal map from the disk with a slit $\{|z| < 1\} \setminus [0, 1]$ to the strip $\{|\mathrm{Re} z| < 1\}$.

9. Let $f$ be a holomorphic function in a complex domain such that $f$ is not identically zero and for any positive integer $n$ there exists a holomorphic function $g$ in the same domain satisfying $g^n = f$. Prove that there exists a holomorphic function $h$ in the same domain such that $e^{ih} = f$.

10. Let $1 < a < \infty$. Prove that the function $e^z - z - a$ has exactly one zero in the left half-plane $\{\mathrm{Re} z < 0\}$.  

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