(1) Let \((\Omega, \mathcal{A}, \mu)\) be a measure space and let \(\{f_n\}\) be a sequence of measurable functions on \(X\). Prove, directly from the definition of convergence almost everywhere, that if \(\sum_n \mu(\{|f_n| > 1/n\}) < \infty\), then the sequence \(\{f_n\}\) converges almost everywhere to zero. Deduce that every sequence of measurable functions that converges in measure to zero has a subsequence that converges almost everywhere to zero.

(2) Show that there is a sequence of nonnegative functions \(\{f_n\}\) in \(L^1(\mathbb{R})\) such that \(\|f_n\|_{L^1(\mathbb{R})} \to 0\), but for any \(x \in \mathbb{R}\), \(\limsup_n f_n(x) = \infty\).

(3) Construct a sequence of nonnegative Lebesgue measurable functions \(\{f_n\}\) on \([0, 1]\) such that
(a) \(f_n \to 0\) almost everywhere, and
(b) for any interval \([a, b] \subseteq [0, 1]\),
\[
\lim_{n \to \infty} \int_a^b f_n(x) dx = (b - a).
\]

(4) In this problem the measure is Lebesgue measure on \([0, 1]\). The norm on \(L^\infty[0, 1]\) is the essential supremum norm, which for a continuous function is the same as the supremum norm.
(a) Prove or disprove that \(L^\infty[0, 1]\) is separable in the norm topology.
(b) Recall that \(L^\infty[0, 1] = (L^1[0, 1])^*\). What is the weak* closure in \(L^\infty[0, 1]\) of the unit ball of \(C[0, 1]\)? Prove your assertion.

(5) Prove that if \(a_1, a_2, \ldots, a_N\) are complex numbers, then
(a) \(\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi i k t)|^p dt \leq \sum_{k=1}^N |a_k|^p\), if \(1 \leq p \leq 2\), and
(b) \(\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi i k t)|^p dt \geq \sum_{k=1}^N |a_k|^p\), if \(2 \leq p < \infty\).

(6) Prove that if \(X\) is an infinite dimensional Banach space and \(X^*\) is separable in the norm topology, then there is a sequence \(\{x_n\}\) of norm one vectors in \(X\) such that \(\{x_n\}\) converges weakly to zero.
(7) Prove or disprove each of the following statements.
   (a) “If \( \{ f_n \} \) is a sequence in \( C[0, 1] \) that converges weakly, then also \( \{ f_n^2 \} \) converges weakly.”
   (b) “If \( \{ f_n \} \) is a sequence in \( L^2[0, 1] \) that converges weakly, then also \( \{ f_n^2 \} \) converges weakly.” (Lebesgue measure on \([0, 1]\).)

(8) Let \( \{ f_n \} \) be a sequence of continuous functions on \( \mathbb{R} \) that converges pointwise to a real valued function \( f \). Prove that for each \( a < b \), the function \( f \) is continuous at some point of \([a, b]\). (Hint: Let \( E_{n, m, k} = \{ f_n - f_m \mid |k| \leq 1/k \}. \))

(9) Let \( X \) and \( Y \) be compact Hausdorff spaces and let \( S \) be the set of all real functions on \( X \times Y \) of the form \( h(x, y) = f(x)g(y) \) with \( f \) in \( C(X) \) and \( g \) in \( C(Y) \). Prove or disprove that the linear span of \( S \) is dense in \( C(X \times Y) \).

(10) Let \( X \) be a Hilbert space and assume that \( \{ x_n \} \) is a sequence in \( X \) that converges weakly to zero. Prove that there is a subsequence \( \{ y_k \} \) of \( \{ x_n \} \) such that the sequence \( \| N^{-1} \sum_{k=1}^{N} y_k \| \) converges to zero. **Caution:** The same statement is NOT true in all Banach spaces; not even in all reflexive Banach spaces.

(11) Let \( F \subset C([0, 1]) \) be a family of continuous functions such that
   1. the derivative \( f'(t) \) exists for all \( t \in (0, 1) \) and \( f \in F \).
   2. \( \sup_{f \in F} |f(0)| < \infty \) and \( \sup_{f \in F} \sup_{t \in (0, 1)} |f'(t)| < \infty \).
   Prove that \( F \) is precompact in the Banach space \( C([0, 1]) \) equipped with the norm \( \| f \| = \sup_{t \in [0, 1]} |f(t)| \).

(12) Let \( \{ x_n \} \) be a weakly Cauchy sequence in a normed linear space \( X \). Prove that
   (a) \( x_n \) is norm bounded in \( X \).
   (b) There exists \( x^* \) in \( X^{**} \) such that \( x_n \) converges weak* to \( x^* \), and \( \| x^* \| \leq \liminf_n \| x_n \| \).