Instructions:

• Read problems very carefully. If you have any questions raise your hand.
• Every question is worth 10 points.
• Justify every non trivial step and give proper citations in your proofs.

1. True or false (prove or give a counter example)

(a) Let \( E \subset \mathbb{R} \) be a Borel set, then \( \{(x, y) \in \mathbb{R}^2 : x - y \in E\} \) is a Borel set in \( \mathbb{R}^2 \).
(b) Let \( E \subset \mathbb{Q} : = [0, 1] \times [0, 1] \). Assume that for every \( x, y \in [0, 1] \) the sets \( E_x = \{y \in [0, 1] : (x, y) \in E\} \) and \( E^y = \{x \in [0, 1] : (x, y) \in E\} \) are Borel. Then \( E \) is Borel.
(c) A function \( f : \mathbb{R} \to \mathbb{R} \) is called Lipschitz if there exists a \( \xi > 0 \) such that \( \forall x, y \in \mathbb{R} \), \( |f(x) - f(y)| \leq \xi |x - y| \). If \( A \subset \mathbb{R} \) is Lebesgue measurable and \( f \) is Lipschitz then \( f(A) \) is Lebesgue measurable.

2. Let \( (X, \mathcal{F}, \mu) \) be a measure space. Is it true that for every measurable essentially bounded \( f : X \to \mathbb{R} \) we have \( \lim_{p \to \infty} \|f\|_p = \|f\|_\infty ? \) Give an answer both in the case that \( \mu \) is finite and the case \( \mu \) is \( \sigma \)-finite.

3. Let \( f : \mathbb{R} \to \mathbb{R} \) Lebesgue integrable and for \( n \in \mathbb{N} \) define

\[ g_n(x) = n \int_{(x, x + \frac{1}{n})} f d\lambda. \]

(a) Prove that \( \lim_{n \to \infty} g_n = f \) \( \lambda \)-a.e.
(b) Pove that for every \( n \in \mathbb{N} \), \( \int_{\mathbb{R}} |g_n| d\lambda \leq \int_{\mathbb{R}} |f| d\lambda. \)
(c) Prove \( \lim_{n \to \infty} \int_{\mathbb{R}} |g_n| d\lambda = \int_{\mathbb{R}} |f| d\lambda. \)

4. Let \( f \in L^1((0, 1]^2, \lambda_2) \) such that \( \int_{(0,x) \times (0,y)} f d\lambda_2 = 0 \) for ever \( x, y \in (0, 1] \). Prove that \( f = 0 \) \( \lambda_2 \)-a.e.
5. Let \( \lambda \) be the Lebesgue measure on \( \mathbb{R} \). Let \( E \subset \mathbb{R} \) be Lebesgue measurable such that \( 0 < \lambda(E) < \infty \). Prove that for all \( 0 \leq \gamma < 1 \) there exists an open interval \( I \subset \mathbb{R} \) such that 
\[
\lambda(E \cap I) \geq \gamma \lambda(I).
\]

6. Let \( X \) be a compact metrizable space and \( \{\mu_n\} \) a sequence of Borel measures on \( X \) with \( \mu_n(X) = 1 \) for every \( n \). Consider the linear map \( \varphi : C(X) \to \ell^\infty(\mathbb{N}) \) defined by \( \varphi(f) = (\int_X f \, d\mu_n)_n \). What conditions on the sequence \( \{\mu_n\} \) are equivalent to \( \varphi \) being an isometry? Provide justification.

7. Let \( X \) be a compact metric space and \( \{f_n\} \) a sequence in \( C(X) \). Prove that \( \{f_n\} \) converges weakly in \( C(X) \) if and only if it converges pointwise and \( \sup_n \|f_n\| < \infty \). Also, give an example of an \( X \) and a sequence \( \{f_n\} \) in \( C(X) \) which converges weakly but not uniformly.

8. Let \( X \) be a Banach space. Show that if \( X^{**} \) is separable then so is \( X \). Also, give an example, with justification, to show that the converse is false.

9. (a) Let \( X \) be a compact metrizable space. Describe the dual of \( C(X) \) according to the Riesz representation theorem.

(b) Consider the spaces \( X = \{1/n : n \in \mathbb{N}\} \cup \{0\} \) and \( Y = [0,1] \) with the topologies inherited from \( \mathbb{R} \). Prove that there does not exist a bijective bounded linear map from \( C(X) \) to \( C(Y) \).

10. Let \( X \) be a Banach space and \( Y \) a subspace of \( X \). Show that \( \|x + Y\| = \inf\{\|x + y\| : y \in Y\} \) defines a norm on \( X/Y \) if and only if \( Y \) is closed.