Texas A&M University

Topology/Geometry Qualifying Exam

August 2017

- There are 8 problems. Work on all of them.
- Prove your assertions.
- Use a separate sheet of paper for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

1. Let X be a metric space with countably many points. Show that X is totally disconnected (by definition, this means that every connected component of X consists of a single point). (Note: countable means finite or countably infinite.)

2. Show that every compact Hausdorff space is normal.

3. (a) Let X be a second countable space and A an uncountable subset of X. Show that A has an accumulation point (a point x in X is an accumulation point of A if every open neighborhood of x contains a point in A other than x).

   (b) Provide an example of a first countable space X and an uncountable subset A of X with no accumulation point.

4. Let X be a topological space and Y a Hausdorff and compactly generated space (by definition, the latter means that a subset C is closed in Y if and only if $C \cap K$ is closed in K for every compact subspace K of Y). Let $f : X \to Y$ be continuous and proper (by definition, the latter means that $f^{-1}(K)$ is compact for every compact subspace K of Y). Show that f is a closed map.

5. (a) Give the definition of an involutive distribution in terms of the vector fields tangent to it.

   (b) Equip $\mathbb{R}^3$ with coordinates (x, y, z) and define two vector fields X and Y by

   $$X = \frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + g(x, y)\frac{\partial}{\partial z}.$$

   Define the distribution $\Delta \subset T\mathbb{R}^3$ by

   $$\Delta = \text{span}(X, Y).$$

   Determine conditions on the functions $f(x, y)$ and $g(x, y)$ that imply Δ is involutive. What do your conditions imply about the maximal connected integral submanifolds of Δ?

6. Let $I \subset \mathbb{R}$ be an interval and define g to be the following Riemannian metric on the surface $I \times \mathbb{S}^1$:

$$g = dr^2 + (f(r))^2 \, d\theta^2,$$

where r is a coordinate on I, θ is a coordinate on $\mathbb{S}^1$, and f is a smooth nonvanishing function.

   (a) Find an orthonormal frame for this metric.

   (b) Find the corresponding dual frame.

   (c) Show that the Gaussian curvature of the surface with this metric is given by $-\frac{f''(r)}{f(r)}$.

7. Consider the following Lie subgroup of $\mathrm{SO}(4)$:

$$\left\{ \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} : a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 = 1 \right\}$$

   Find its Lie algebra.

8. (a) Let $\mathrm{GL}(2, \mathbb{R})$ denote the space of 2×2 matrices with real entries and nonvanishing determinant. Show that $\mathrm{GL}(2, \mathbb{R})$ is a manifold. What is its dimension?

   (b) Let $\det : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}$ denote the determinant function. Show that 1 is a regular value of det.