• There are 10 problems. Work on all of them and prove your assertions.

• Use a separate sheet of paper for each problem and write only on one side of the paper.

• Write your name on the top right corner of each page.

1. Let $X$ be a compact metric space. Show that if $f : X \rightarrow X$ satisfies $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$ (i.e., if $f$ is an isometry) then $f$ is a homeomorphism.

2. Prove that the metric space $X$ is complete if and only if for every sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ of nonempty closed subsets of $X$ such that diameters of $A_n$ converge to 0, the intersection $\bigcap_{i=1}^{\infty} A_i$ is non-empty.

3. Let $X_i$, for $i \in I$, be a family of topological spaces, and let $A_i \subset X_i$ be subsets. Show that $\prod_{i \in I} A_i = \prod_{i \in I} \overline{A_i}$, where closure on the left-hand side of the equality is taken with respect to the product topology on $\prod_{i \in I} X_i$.

4. Let $X$ and $Y$ be topological spaces, where $Y$ is compact. Let $p : X \times Y \rightarrow X$ be the projection onto the first factor. Show that $p$ is closed (i.e., maps each closed subset of $X \times Y$ to a closed subset of $X$).

5. Show that any map $f : S^1 \rightarrow S^1$ of degree 1 is homotopic to the identity.

6. (a) Given a differential $p$-form $\omega$ on a manifold $N$ and a smooth map $g : M \rightarrow N$ give the definition of the pull-back $g^* \omega$ of the form $\omega$ by the map $g$.

(b) Define $g : \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\} \rightarrow \mathbb{R}^3 \setminus \{0\}$ by $(x, y, z) = g(u, v) = (u, v, \sqrt{1-u^2-v^2})$ and

$$\omega = \frac{x dy \wedge dz + ydz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$ 

Compute $g^* \omega$ and $d \omega$ and verify by direct computations that $g^*(d \omega) = d(g^* \omega)$

(c) Using the calculations of $g^* \omega$ from the previous item, calculate $\int_S \omega$, where $S$ is the upper unit hemisphere in $\mathbb{R}^3$, i.e. $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$.

7. (a) Given a smooth map $F : M \rightarrow N$ between two smooth manifolds $M$ and $N$ define the notions of a critical point and a critical value of $F$.

(b) Define $Z := \{(x, p, q) \in \mathbb{R}^3 : x^3 + px + q = 0\}$.

i. Prove that $Z$ is a smooth submanifold of $\mathbb{R}^3$;

ii. Define $\pi : Z \rightarrow \mathbb{R}^2$ by $\pi(x, p, q) = (p, q)$ for every $(x, p, q) \in Z$. Prove that $(p, q)$ is a critical value of $\pi$ if and only of $4p^3 + 27q^2 = 0$. 

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8. Let \( H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\} \) be the upper half plane with the Riemannian metric \( g = \frac{dx^2 + dy^2}{y^2} \).
Calculate the Gaussian curvature of this metric.

9. (a) Let \( S \) be a smooth tensor field of type \((r, s)\) on a smooth manifold \( M \) and \( X \) be a smooth vector field on \( M \). Give the definition of the Lie derivative \( L_X S \) of the tensor field \( S \) with respect to the vector field \( X \) (Here the definition, which uses certain limit and does not involve Lie brackets, is expected).

(b) Prove that if \( X \) and \( Y \) are two smooth vector fields on \( M \), then \( L_X Y = [X, Y] \), where \([X, Y]\) is the Lie bracket (the commutator) of \( X \) and \( Y \).

(c) Assume that vector fields \( X \) and \( Y \) commute and linearly independent in a neighborhood of point \( p_0 \) in \( M \), i.e., \([X, Y](p) = 0\) and the dimension of \( \text{span}(X(p), Y(p)) \) is equal to 2 for every \( p \) in this neighborhood. Prove that there is a coordinate system \((U, x_1, \ldots, x_n)\) around \( p_0 \) (here \( n = \text{dim} M \)) such that \( X = \frac{\partial}{\partial x_1} \) and \( Y = \frac{\partial}{\partial x_2} \) on \( U \).

10. (a) Assume that \((\omega_1, \ldots, \omega_k)\) is a collection of independent 1-forms defining the distribution \( D \) in an open set \( U \) of \( M \), i.e. \( D(p) = \{X \in T_p M : \omega_1(X) = \ldots = \omega_k(X) = 0\} \) for any \( p \in U \). Describe the involutivity of \( D \) in terms of the forms \( \omega_i \).

(b) Let \( G \) be a Lie group and \( g \) be the corresponding Lie algebra. Recall that the Maurer-Cartan form \( \Omega \) on \( G \) is \( g \)-valued 1-form satisfying \( \Omega_g(v) = (L_{g^{-1}})_* v \) for every \( g \in G \) and \( v \in T_g G \), where \( L_g \) denotes the left translation by \( g \) in \( G \). Prove that \( \Omega \) satisfies
\[
\frac{d}{d\lambda} \Omega(g, \lambda) = -[\Omega(g), \Omega(\lambda)],
\]
where in the right-hand side \([\cdot, \cdot]\) means the brackets in the Lie algebra \( g \).

(c) Here we use the notations of the previous item. Let \( M \) be a smooth manifold endowed with a \( g \)-valued 1-form \( \Phi \) satisfying \( d\Phi(X, Y) + [\Phi(X), \Phi(Y)] = 0 \). Prove that for any \( p \in M \) there exists a neighborhood \( U \) of \( p \) and a smooth map \( F : U \rightarrow G \) such that \( \Phi = F^* \Omega \). (Hint: Consider an appropriate involutive distribution on \( M \times G \) such that the graph of the required map \( F \) is an integral submanifold of this distribution).