This exam has 8 questions, for a total of 120 points.

Please answer each question in the space provided. You need to write full solutions. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

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Problem 1. (15 pts)  

(a) Let $X$ be a Hausdorff topological space. If $\{x_n\}$ is a convergent sequence in $X$, prove that $\lim_{n \to \infty} x_n$ is unique.

(b) If the Hausdorff condition in part 1 is dropped. Is the same conclusion still true? Prove your claim if your answer is yes. Give a counter example if your answer is no.
Problem 2. (15 pts)
Let the real line $\mathbb{R}$ be given the standard topology. Construct a topology on $\mathbb{R}^2$ such that every function from $\mathbb{R}^2$ to $\mathbb{R}$ is continuous. Justify your answer.
Problem 3. (15 pts)
Let $X$ be the connected sum of the torus with the Klein bottle. Compute the fundamental group of $X$. 
Problem 4. (15 pts)
Let $D^2$ be the closed disk in the plane. Prove that any continuous map $f : D^2 \to D^2$, has a fixed point, i.e. there exists $p \in D^2$ satisfying $f(p) = p$. 
Problem 5. (15 pts)

(a) Suppose $M$ is a smooth $n$-dimensional manifold and $p \in M$. Define the tangent space $T_pM$ of $M$ at $p$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $f(x, y) = (x, xy)$. Compute the pushforward $f_*(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$.

(c) Determine the regular values of the map $f$ in the previous part.
Problem 6. (15 pts)

Let $O(n)$ denote the set of $n \times n$ matrices $A$ so that $AA^\top = I$, where $I$ is the $n \times n$ identity matrix.

(a) Show that $O(n)$ is a Lie group.

(b) Find the Lie algebra of $O(n)$ and determine its dimension.
Problem 7. (15 pts)

(a) Let $\Delta$ be the distribution $\Delta = \ker(dz - y\,dx)$ on $\mathbb{R}^3$, i.e., $\vec{v} \in \Delta_{(x,y,z)}$ if $(dz - y\,dx)(\vec{v}) = 0$. Determine whether $\Delta$ is an involutive distribution.

(b) Let $\alpha$ be a 1-form on $\mathbb{R}^3$ and suppose that $\Delta = \ker(\alpha)$ is a distribution on $\mathbb{R}^3$. Suppose further that $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$. Show that $\Delta$ is never involutive.
Problem 8. (15 pts)

Let $S \subset \mathbb{R}^3$ be the paraboloid

$$S = \{z = x^2 + y^2\}.$$

(a) Let $g$ be the metric on $S$ induced by the standard metric on $\mathbb{R}^3$. Show that $g$ has the form

$$g = (1 + 4r^2)dr^2 + r^2d\theta^2,$$

where $(r, \theta)$ are polar coordinates in $(x, y)$.

(b) Find an orthonormal frame for this metric and the corresponding dual coframe.

(c) Show that the Gaussian curvature of $S$ with this metric is $4/(1 + 4r^2)^2$. 