TEXAS A&M UNIVERSITY

TOPOLOGY/GEOMETRY QUALIFYING EXAM

January 2019

- `There are 8 problems. Work on all of them and prove your assertions.`
- `Use a separate sheet of paper for each problem and write only on one side of the paper.`
- `Write your name on the top right corner of each page.`

1. Suppose that X and Y are connected spaces, and $A \subset X$ and $B \subset Y$ are proper subsets. Prove that the space $(X \times Y) \setminus (A \times B)$ is connected.

2. Prove that none of the following spaces are homeomorphic to each other $\mathbb{R}^2, S^1 \times \mathbb{R}, S^2, S^1 \times S^1, \mathbb{R}^3, S^3$.

3. Prove that any continuous map from the real projective plane to the 2-dimensional torus $S^1 \times S^1$ is null-homotopic.

4. Prove that if X is Hausdorff and Y is a retract of X, then Y is closed in X.

5. (a) Formulate the Implicit Function Theorem.

   (b) Assume that M is a smooth manifold and a group such that the group operation is a smooth map (from $M \times M$ to M). Prove that the operation of taking inverse is a smooth map (from M to itself).

   (c) For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by $M_a = \{(x, y) \in \mathbb{R}^2 : y^2 = x(x-1)(x-a)\}$. For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$?

6. (a) Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ be the 2-torus, defined by $w^2 + x^2 = y^2 + z^2 = 1$, with the orientation determined by its product structure. Compute $\int_{\mathbb{T}^2} wy\, dx \wedge dz$.

   (b) Let M be an oriented smooth compact n-dimensional manifold with boundary ∂M and suppose that ∂M has two connected components N_0 and N_1. Let $\imath_j : N_j \to M$ be the inclusion map for $j = 0, 1$. Suppose that α is a p-form with $\imath_0^* \alpha = 0$ and β is an $(n-p-1)$-form with $\imath_1^* \beta = 0$. Prove that in this case $\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$.

   (c) Let ω be a *closed* 1-form on $\mathbb{R}^2 \backslash \{0\}$ and $\alpha = -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$. Show that there exist a constant c and a smooth function $g : R^2 \backslash \{0\} \mapsto \mathbb{R}$ such that $\omega = c\,\alpha + d\,g$.

7. (a) Let $M = \mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4). Let D be the distribution in $\mathbb{R}^4$ defined by the following two 1-forms: $\omega_1 = dx_1 - x_2 dx_4, \quad \omega_2 = dx_2 - x_3 dx_4$. Is this distribution involutive? Prove your answer.

(b) Let M and N be 2-dimensional smooth manifolds. Let (α_1, α_2) be a coframe on M and (ω_1, ω_2) be a coframe on N such that there are constants k_1 and k_2 so that

$$d\alpha_1 = k_1\, \alpha_1 \wedge \alpha_2, \qquad d\alpha_2 = k_2\, \alpha_1 \wedge \alpha_2, \qquad d\omega_1 = k_1\, \omega_1 \wedge \omega_2, \qquad d\omega_2 = k_2\, \omega_1 \wedge \omega_2.$$

Prove that for every $q \in M$ and $p \in N$ there exist a neighborhood U of q in M, a neighborhood V of p in N, and a diffeomorphism $F : U \to V$ such that $\alpha_1 = F^*\omega_1, \quad \alpha_2 = F^*\omega_2$. in U.

Hint: Let $\pi : M \times N \to M$ and $p : M \times N \to N$ are canonical projections. Work with the distribution on $M \times N$ defined by 1-forms $\pi^*\alpha_1 - p^*\omega_1$ and $\pi^*\alpha_2 - p^*\omega_2$.

8. Set $N = (0, 1) \in \mathbb{S}^1 \subset \mathbb{R}^2$ and $U = \mathbb{S}^1 \backslash N$. Let (U, φ) be a smooth chart, where $\varphi : U \to \mathbb{R}$ is given by the stereographic projection $\varphi(x, y) = \dfrac{x}{1-y}$. Let t be the standard Cartesian coordinate on $\mathbb{R}$.

(a) Assume that $X_1 = t^2 \frac{\partial}{\partial t}$ is the vector field on $\mathbb{R}$ and Y_1 is the vector field on U such that Y_1 and X_1 are φ-related, i.e. $X_1 = \varphi_* Y_1$. Does Y_1 extend to a smooth vector field on $\mathbb{S}^1$? Justify your answer.

(b) Assume that $X_2 = t^3 \frac{\partial}{\partial t}$ is the vector field on $\mathbb{R}$ and Y_2 is the vector field on U such that Y_2 and X_2 are φ-related. Does Y_2 extend to a smooth vector field on $\mathbb{S}^1$? Justify your answer.