1. Suppose that $X$ and $Y$ are connected spaces, and $A \subset X$ and $B \subset Y$ are proper subsets. Prove that the space $(X \times Y) \setminus (A \times B)$ is connected.

2. Prove that none of the following spaces are homeomorphic to each other: $\mathbb{R}^2$, $S^1 \times \mathbb{R}$, $S^2$, $S^1 \times S^1$, $\mathbb{R}^3$, $S^3$.

3. Prove that any continuous map from the real projective plane to the 2-dimensional torus $S^1 \times S^1$ is null-homotopic.

4. Prove that if $X$ is Hausdorff and $Y$ is a retract of $X$, then $Y$ is closed in $X$.

5. (a) Formulate the Implicit Function Theorem.
    (b) Assume that $M$ is a smooth manifold and a group such that the group operation is a smooth map (from $M \times M$ to $M$). Prove that the operation of taking inverse is a smooth map (from $M$ to itself).
    (c) For each $a \in \mathbb{R}$, let $M_a$ be the subset of $\mathbb{R}^2$ defined by $M_a = \{(x,y) \in \mathbb{R}^2 : y^2 = x(x-1)(x-a)\}$. For which values of $a$ is $M_a$ an embedded submanifold of $\mathbb{R}^2$?

6. (a) Let $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$ be the 2-torus, defined by $w^2 + x^2 = y^2 + z^2 = 1$, with the orientation determined by its product structure. Compute $\int_{T^2} wy \, dx \wedge dz$.
    (b) Let $M$ be an oriented smooth compact $n$-dimensional manifold with boundary $\partial M$ and suppose that $\partial M$ has two connected components $N_0$ and $N_1$. Let $i_j : N_j \to M$ be the inclusion map for $j = 0, 1$. Suppose that $\alpha$ is a $p$-form with $i_0^*\alpha = 0$ and $\beta$ is an $(n-p-1)$-form with $i_1^*\beta = 0$. Prove that in this case $\int_M \alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$.
    (c) Let $\omega$ be a closed 1-form on $\mathbb{R}^2 \setminus \{0\}$ and $\alpha = -\frac{y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy$. Show that there exist a constant $c$ and a smooth function $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $\omega = c \alpha + dg$.

7. (a) Let $M = \mathbb{R}^4$ with coordinates $(x_1, x_2, x_3, x_4)$. Let $D$ be the distribution in $\mathbb{R}^4$ defined by the following two 1-forms: $\omega_1 = dx_1 - x_2 dx_4$, $\omega_2 = dx_2 - x_3 dx_4$. Is this distribution involutive? Prove your answer.
(b) Let $M$ and $N$ be 2-dimensional smooth manifolds. Let $(\alpha_1, \alpha_2)$ be a coframe on $M$ and $(\omega_1, \omega_2)$ be a coframe on $N$ such that there are constants $k_1$ and $k_2$ so that
\[
d\alpha_1 = k_1 \alpha_1 \wedge \alpha_2, \quad d\alpha_2 = k_2 \alpha_1 \wedge \alpha_2, \quad d\omega_1 = k_1 \omega_1 \wedge \omega_2, \quad d\omega_2 = k_2 \omega_1 \wedge \omega_2.
\]
Prove that for every $q \in M$ and $p \in N$ there exist a neighborhood $U$ of $q$ in $M$, a neighborhood $V$ of $p$ in $N$, and a diffeomorphism $F : U \to V$ such that $\alpha_1 = F^*\omega_1$, $\alpha_2 = F^*\omega_2$. in $U$.

Hint: Let $\pi : M \times N \to M$ and $p : M \times N \to N$ are canonical projections. Work with the distribution on $M \times N$ defined by 1-forms $\pi^*\alpha_1 - p^*\omega_1$ and $\pi^*\alpha_2 - p^*\omega_2$.

8. Set $N = (0, 1) \in S^1 \subset \mathbb{R}^2$ and $U = S^1 \setminus N$. Let $(U, \varphi)$ be a smooth chart, where $\varphi : U \to \mathbb{R}$ is given by the stereographic projection $\varphi(x, y) = \frac{x}{1 - y}$. Let $t$ be the standard Cartesian coordinate on $\mathbb{R}$.

(a) Assume that $X_1 = t^2 \frac{\partial}{\partial t}$ is the vector field on $\mathbb{R}$ and $Y_1$ is the vector field on $U$ such that $Y_1$ and $X_1$ are $\varphi$-related, i.e. $X_1 = \varphi_* Y_1$. Does $Y_1$ extend to a smooth vector field on $S^1$? Justify your answer.

(b) Assume that $X_2 = t^3 \frac{\partial}{\partial t}$ is the vector field on $\mathbb{R}$ and $Y_2$ is the vector field on $U$ such that $Y_2$ and $X_2$ are $\varphi$-related. Does $Y_2$ extend to a smooth vector field on $S^1$? Justify your answer.