TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
January 2020

- `There are 10 problems. Work on all of them and prove your assertions.`
- `Use a separate sheet for each problem and write only on one side of the paper.`
- `Write your name on the top right corner of each page.`

Q1. Let Y be a *locally connected space* and let $A \subset Y$ be an arbitrary subspace. Show that if the boundary ∂A of A is locally connected then its closure $\overline{A}$ is also locally connected.

Q2. Let $A \subset X$ be a closed subspace of a *normal* space X and let $f\colon A \to S^n$ be a continuous map, where S^n is the unit sphere of dimension n. Show that there exists an open neighborhood $U \supset A$ of A in X over which f can be extended to a map $F\colon U \to S^n$.

Q3. Let $S^1 \subset \mathbb{R}^2$ be the unit circle centered at $\mathbf{0} = (0,0)$ and let $\mathbf{1} = (1,0)$ be its canonical base point. Give the natural numbers $\mathbb{N}$ the discrete topology and define the quotient space $X := (S^1 \times \mathbb{N})/(\{\mathbf{1}\} \times \mathbb{N})$. Now, let $K := \cup_{n\in\mathbb{N}} C_n$ be the subspace of $\mathbb{R}^2$ obtained as the union of circles $C_n := \{(x,y) \in \mathbb{R}^2 \mid (x+1/n)^2 + y^2 = 1/n^2\}$ centered at $(-1/n, 0)$ and having radius $1/n$, for all $n \in \mathbb{N}$.

a. Prove that X and K are *metrizable spaces*.
b. Exhibit a continuous bijection $f\colon X \to K$.
c. Are X and K *homeomorphic*? Explain your answer.

Q4. Let J be an uncountable set and let $X := \mathcal{P}(J)$ denote its power set. Given finite (possibly empty) disjoint subsets $A, B \subset J$, define

$$\mathcal{N}_{A,B} := \{C \in X \mid A \cap C = \emptyset \text{ and } B \subseteq C\}.$$

a. Show that the collection $\mathcal{B} := \{\mathcal{N}_{A,B} \mid A, B \subset J \text{ are finite, and } A \cap B = \emptyset\}$ forms a *basis for a topology* on X.
b. Show that X is a **compact Hausdorff** space with this topology.
c. Let $\mathcal{K} := \{C \subset J \mid C \text{ is at most countable}\}$. Show that $\mathcal{K}$ is a *dense* and *completely regular* subspace of X.
d. Show that if $H \subset \mathcal{K}$ is a countable subset then $\overline{H} \subset \mathcal{K}$.

Q5. A map $p\colon X \to Y$ is called *perfect* if it is a continuous, closed and surjective map, such that $p^{-1}(\{y\})$ is compact for each $y \in Y$.

a. Let $p\colon X \to Y$ be a perfect map between Hausdorff spaces X and Y. Show that X is *paracompact* if and only if Y is paracompact.
b. Show that the product of a compact space and a paracompact space is paracompact.

Q6. a. Formulate the Implicit Function Theorem.

b. Let $P_t(x) = a_n(t)x^n + a_{n-1}(t)x^{n-1} + \ldots + a_0(t), \quad T \in R$ be a smooth family of polynomials over $\mathbb{R}$, i.e. all coefficients $a_i(t)$ are smooth in t. Assume that x_0 is a simple (i.e., not repeated root) of the polynomial P_0.

(i) Prove that there exists $\varepsilon > 0$ and smooth function $x : (-\varepsilon, \varepsilon) \mapsto \mathbb{R}$ such that $x(t)$ is a root of P_t for every $t \in (-\varepsilon, \varepsilon)$ and $x(0) = 0$.

(ii) Is the statement of the previous item always true, if x_0 is a root of multiplicity 2 of $P_0(x)$? Justify our answer.

Q7. a. Let V be a 4-dimensional vector space. Is there a $\omega \in \bigwedge^2 V^*$ such that the restriction of ω to every 2-dimensional subspace $W \subset V$ is nonzero? Prove your answer.

b. Let M be an even dimensional manifold and ω be a closed nondegenerate differential 2-form (nondegenerate means that for any $m \in M$ if $i_X\omega|_p = 0$ for some $X \in T_mM$, then $X = 0$).

(i) Prove that for any smooth function h on M there is a unique vector field $\vec{h}$ such that $i_{\vec{h}}\omega = -dh$.

(ii) Prove that for any smooth function h on M we have that $L_{\vec{h}}\omega = 0$ and $(e^{t\vec{h}})^*\omega = \omega$, where $e^{t\vec{h}}$ denotes the local flow generated by $\vec{h}$.

Q8. a. Define what is an *involutive distribution* and *an integral manifold* of a distribution.

b. Prove that the *existence* of an integral manifold to a distribution D through any point implies that D is involutive.

c. In $\mathbb{R}^3$, set $X_1 = x_1^2 x_2 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}$, $\quad X_2 = 2x_1 \frac{\partial}{\partial x_2}$.

(i) Prove that for any point $p \in \mathbb{R}^3$ there are no neighborhood U and coordinate functions y_1, y_2, y_3 on U such that $X_1 = \frac{\partial}{\partial y_1}$ and $X_2 = \frac{\partial}{\partial y_2}$.

(ii) Define the distribution $D = \operatorname{span}(X_1, X_2)$ on $M := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \neq 0\}$. Prove that for any $p \in M$ there exist a neighborhood U, coordinate functions (y^1, y^2, y^3) on U, and vector fields Y_1 and Y_2 on U so that $D = \operatorname{span}(Y_1, Y_2)$ and $Y_i = \frac{\partial}{\partial y_i}$, $i = 1, 2$.

(iii) Give an example of such vector fields Y_1, Y_2 (in the original coordinates (x_1, x_2, x_3)).

Q9. Let X_1, X_2 be a frame in $\mathbb{R}^2$, i.e. X_1 and X_2 are smooth vector fields and for every $\mathbf{x} \in \mathbb{R}^2$ the tuple $\big(X_1(\mathbf{x}), X_2(\mathbf{x})\big)$ form a basis of $T_{\mathbf{x}}\mathbb{R}^2$. Let (ω^1, ω^2) be the dual coframe to (X_1, X_2), i.e. ω^i, $i = 1, 2$, are 1-forms such that for every $\mathbf{x} \in \mathbb{R}^2$ we have $\omega^i|_{\mathbf{x}}(X_j(\mathbf{x})) = \delta_i^j$.

a. Prove that there exists the unique 1-form ω_2^1 such that

$$d\omega^1 = \omega_2^1 \wedge \omega^2, \qquad d\omega^2 = \omega_2^1 \wedge \omega^1$$

The form ω_2^1 is called the *canonical connection form* of the frame (X_1, X_2).

b. Let $(\tilde{X}_1, \tilde{X}_2)$ be another frame in R^2 such that for a smooth function $\theta : \mathbb{R}^2 \to \mathbb{R}$ with

$$\widetilde{X}_1 = \cosh\big(\theta(\mathbf{x})\big)X_1 + \sinh\big(\theta(\mathbf{x})\big)X_2, \quad \widetilde{X}_2 = \sinh\big(\theta(\mathbf{x})\big)X_1 + \cosh\big(\theta(\mathbf{x})\big)X_2,$$

where cosh and sinh are hyperbolic cosine and sine, and let $\widetilde{\omega}_2^1$ be the canonical connection form of the frame $(\widetilde{X}_1, \widetilde{X}_2)$. Prove that $d\widetilde{\omega}_2^1 = d\omega_2^1$.

c. Let K and $\widetilde{K}$ are to functions on $\mathbb{R}^2$ such that

$$d\omega_2^1 = K\omega^1 \wedge \omega^2, \quad d\widetilde{\omega}_2^1 = \widetilde{K}\widetilde{\omega}^1 \wedge \widetilde{\omega}^2.$$

Prove that $\widetilde{K} \equiv K$.

Q10. a. Find the Gaussian curvature of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ in $\mathbb{R}^3$ at the vertices (i.e. at the points where two of the coordinates x, y, and z vanish).

b. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that $f(x, y) = 0$ for all (x, y) outside the unit disk, i.e., for all (x, y) with $x^2 + y^2 \geq 1$. Consider the surface S in $\mathbb{R}^3$ given by the graph of f over the disk $x^2 + y^2 \leq 2$. What can you say about the integral of the Gaussian curvature over S? Prove your answer.

Hint: Use the Gauss-Bonnet theorem