TEXAS A&M UNIVERSITY
TOPOLOGY/GEOMETRY QUALIFYING EXAM
January 2022

- There are 10 problems. Work on all of them and prove your assertions.
- Use a separate sheet for each problem and write only on one side of the paper.
- Write your name on the top right corner of each page.

Q1. Let (X, d) be a metric space. Suppose A is nonempty compact subset of X.

(a) Define the distance between of a point $x \in X$ and A to be

$$d(x, A) := \inf\{d(x, a) \mid a \in A\}.$$

For a given $x \in X$, prove that $d(x, A) = d(x, a_0)$ for some $a_0 \in A$.

(b) Define the diameter of A to be

$$\operatorname{diam}(A) := \sup\{d(x, y) \mid x, y \in A\}.$$

Show that $\operatorname{diam}(A)$ is finite and there exist $x_0, y_0 \in A$ such that

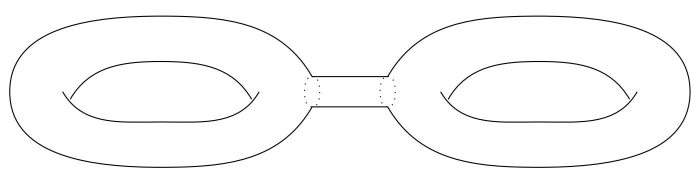
$$\operatorname{diam}(A) = d(x_0, y_0).$$

(c) Suppose B is a closed subset of X such that $A \cap B = \emptyset$. Define the distance between A and B to be

$$d(A, B) := \inf_{b \in B} d(b, A).$$

Prove that $d(A, B) > 0$.

Q2. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-dimensional torus. Compute the fundamental group of the connected sum of $\mathbb{T}^2$ with $\mathbb{T}^2$ (namely a closed oriented surface of genus two). Here the connected sum means the following. We remove a small disc from each of the tori and glue the resulting spaces along the common boundary (which is a circle) to obtain a new topological space called the connected sum (see the picture).



Q3. Let $\widetilde{X}$ and $\widetilde{Y}$ be simply connected covering spaces of the path-connected, locally path-connected spaces X and Y respectively. Show that if X is homotopy equivalent to Y, then $\widetilde{X}$ is homotopy equivalent to $\widetilde{Y}$.

Q4. Let X be a Hausdorff space and A a compact subset of X. Let X/A be the quotient space obtained by identifying A to a single point. Show that X/A is Hausdorff (with respect to the quotient topology).

Q5. Let X and Y be two topological spaces. Suppose Y is Hausdorff. Show that if $f, g \colon X \to Y$ are two continuous maps such that $f|_A = g|_A$ on some dense subset $A \subset X$, then $f = g$ on X.

Q6. Inside the plane $\mathbb{R}^2$, let L be the union of the positive x-axis and the positive y-axis (including the origin), i.e.,

$$L = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0,\ xy = 0\}.$$

(a) Prove that L is a topological manifold.

(b) Prove that L is not a smooth submanifold of $\mathbb{R}^2$. (Hint: L is a smooth submanifold if and only if locally L is defined by $f(x, y) = 0$ for some smooth function f with df nonvanishing.)

Q7. Let n be a positive integer. Consider the set of orthogonal $n \times n$ matrices $O(n)$ as a subset of the set of all $n \times n$ real matrices $\mathrm{Mat}_{n\times n}$ (which can be identified with the Euclidean space $\mathbb{R}^{n^2}$). Prove that $O(n)$ is an embedded submanifold of $\mathrm{Mat}_{n\times n}$. (Hint: realize $O(n)$ as the preimage of a smooth map from $\mathrm{Mat}_{n\times n}$ to another Euclidean space at a regular value and use the implicit function theorem.)

Q8. Let M be a smooth manifold and α is a differential 1-form. Prove that for any two vector fields X, Y on M, one has

$$(d\alpha)(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

Here $X(\alpha(Y))$ is the directional derivative of the function $\alpha(Y)$ in the direction X and $[X, Y]$ is the Lie bracket.

Q9. Let $S \subset \mathbb{R}^3$ be a regular, connected, compact, orientable surface (without boundary) which is not homeomorphic to a sphere. Let $S_+, S_0, S_- \subset S$ be the set of points on S where the Gauss curvature is positive, zero, and negative, respectively. Prove that the three sets are all nonempty.

Q10. Let M be a smooth manifold and $\alpha_1, \ldots, \alpha_k$ are closed differential forms on M, i.e, $d\alpha_i = 0$ for all $i = 1, \ldots, k$.

(a) Prove that $\alpha_1 \wedge \cdots \wedge \alpha_k$ is closed.

(b) If one of $\alpha_1, \ldots, \alpha_k$ is an exact differential form, prove that $\alpha_1 \wedge \cdots \wedge \alpha_k$ is exact.