Real analysis qualifying exam

January 2024

Each problem is worth ten points. Work each problem on a separate piece of paper. If you are not sure whether or not you are allowed to use a particular result to solve a problem, ask. m denotes the Lebesgue measure.

1. Let E be a Lebesgue measurable set of positive measure. Show that for any $0 < \alpha < 1$, there is an open interval I such that $m(E \cap I) > \alpha\, m(I)$.

2. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $[0,1]$, and $(c_n)_{n=1}^{\infty}$ be a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} c_n < \infty$. Show that the series

$$\sum_{n=1}^{\infty} \frac{c_n}{|x - x_n|^{1/2}}$$

converges for almost every $x \in [0,1]$.

3. Find the sum

$$\sum_{k=2}^{\infty} (-1)^k \sum_{n=2}^{\infty} \frac{1}{n^k}.$$

Justify your calculation.

4. Let f be a Lebesgue measurable function on $[0,1]$ such that $f > 0$ a.e. Suppose $(E_n)_{n=1}^{\infty}$ is a sequence of measurable sets with the property that

$$\int_{E_n} f\, dx \to 0.$$

Prove that $m(E_n) \to 0$.

5. Recall that a point x is called isolated if $\{x\}$ is an open set. Show that a compact metric space with no isolated points is uncountable.

6. Recall that the graph of a function $f : X \to Y$ is the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$.

(a) State the Closed Graph Theorem.
(b) Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ whose graph is closed. Here $\mathbb{R}$ has standard topology.
(c) Give an example of a discontinuous linear function $f : X \to Y$, where X and Y are both normed spaces, whose graph is closed. Justify your answer.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space with μ a probability measure. Recall that

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}, \quad \|f\|_\infty = \operatorname{esssup} |f| .$$

Show that $\|f\|_p$ is an increasing function of p for $0 < p \le \infty$.

8. Let X and Y be reflexive Banach spaces such that

- Y^* is separable.
- There exists a continuous linear transformation T from X to Y with kernel $\{0\}$.

Prove that X^* is separable.

9. Let $\mathcal{P}$ be the space of real-valued polynomials, and $\mathcal{P}_n$ the subspace of polynomials of degree at most n. Fix $a \in \mathbb{R}$.

(a) Show that for every n, there exists a unique $g_n \in \mathcal{P}_n$ such that for all $f \in \mathcal{P}_n$,

$$f(a) = \int_0^1 f(x) g_n(x)\, dx.$$

(b) Show that there does not exist a Lebesgue integrable $h \in L^1([0,1], dx)$ such that for all $f \in \mathcal{P}$,

$$f(a) = \int_0^1 f(x) h(x)\, dx.$$

10. Let $C[0,1]$ be the space of all real-valued continuous functions on $[0,1]$. For $f \in C[0,1]$, denote $co(f)$ the smallest closed convex subset of $\mathbb{R}$ containing $\{f(x) : 0 \leq x \leq 1\}$. Let Φ be a linear mapping from $C[0,1]$ to $\mathbb{R}$ such that $\Phi(f) \in co(f)$ for each f. Prove that

$$\lim_{n\to\infty} \Phi\left(\frac{n^2}{(nx-1)^2 + n^2}\right) = \Phi\left(\frac{1}{x^2+1}\right).$$