Real analysis qualifying exam

August 2024

Each problem is worth ten points. Work each problem on a separate piece of paper. If you are not sure whether or not you are allowed to use a particular result to solve a problem, ask. dx denotes the Lebesgue measure.

1. Let $f \in L^+(X, \mathcal{M}, \mu)$ be a non-negative measurable function on a measure space.

(a) Show that if $\int f \, d\mu < \infty$, then $f < \infty$ a.e.
(b) Show that if $\int f \, d\mu = 0$, then $f = 0$ a.e.

2. Let $f(x,t) : [0,1] \times [0,1] \to \mathbb{R}$ be a measurable function such that for every $x \in [0,1]$, the mapping $t \mapsto f(x,t)$ is continuous, and furthermore there exists a function $g \in L^1([0,1], dx)$ such that for each $(x,t) \in [0,1] \times [0,1]$, $|f(x,t)| \leq g(x)$. Show that

$$h(t) = \int_0^1 f(x,t) \, dx$$

is continuous.

3. Let X and Y be topological spaces, and $X \times Y$ their product space with the product topology. Denote $\mathcal{B}_X$, $\mathcal{B}_Y$, $\mathcal{B}_{X \times Y}$ the corresponding Borel σ-algebras. Show that if $A \in \mathcal{B}_X$ and $B \in \mathcal{B}_Y$, then $A \times B \in \mathcal{B}_{X \times Y}$.

4. Let F be a Lipschitz continuous function on $\mathbb{R}$, that is, $\left| \frac{F(x) - F(y)}{x - y} \right| \leq M$ for all $x \neq y$. Recall that this implies that F' exists a.e. on $\mathbb{R}$. Show that for any $a < b$,

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

You are not allowed to refer to the fact that this conclusion holds for any absolutely continuous function.

5. Let

$$C_{0,0}[0,1] = \{ f \in C([0,1], \mathbb{R}) : f(0) = f(1) \}.$$

Let $P_{0,0}$ be the subset of polynomials in $C_{0,0}[0,1]$. Prove that $P_{0,0}$ is dense in $C_{0,0}[0,1]$ in the uniform topology.

6. Show that a normed space X is complete if and only if any absolutely convergent series is convergent (that is, whenever $\sum \|x_n\| < \infty$, the series $\sum x_n$ converges in X).

7. Let X and Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for every $f \in Y^*$, show that T is bounded.

8. Let X be a Banach space, $V \subset X$ a closed subspace, and $x \in X \setminus V$.

(a) Prove that there exists a linear functional $\phi_{x,V} \in X^*$ such that $\phi_{x,V}|_V = 0$, $\|\phi_{x,V}\| = 1$, and $\phi_{x,V}(x) = \inf_{y \in V} \|x - y\|$.
(b) Suppose X is a Hilbert space, and V has an orthonormal basis $\{v_i : i \in I\}$. Find a formula for $\phi_{x,V}$.

9. Let $(X, \mathcal{M}, \mu)$ be a *finite* measure space, and $1 < p < \infty$. Let $f, f_n \in L^p(X, d\mu)$ for $n \in \mathbb{N}$ be functions such that $f_n \to f$ pointwise a.e. and $\sup_n \|f_n\|_p < \infty$. Show that $f_n \to f$ weakly. You may use without proof that an integrable function is uniformly integrable. Comment: the result holds in general measure spaces, but you are not asked to prove that.

10. Give examples of the following. Justify your answers.

(a) A Banach space X, a closed subspace V, and a point $x \in X$ such that
$$\|x - y\| = \inf_{z \in V} \|x - z\|$$
for multiple $y \in V$.
(b) A bounded linear bijection between normed spaces which is not a homeomorphism.
(c) A bounded linear functional on ℓ^∞ which does not arise from duality with ℓ^1.