We will write the base ten expansion of a number \( n \) by \( a_k a_{k-1} \ldots a_1 a_0 \), so that \( 0 \leq a_i \leq 9 \) and \( n = \sum_{i=0}^{k} a_i 10^i \).

Let us prove the following simple properties of \( S(n) \), which will be used later.

**Lemma 1.** The difference \( S(n) - n \) is divisible by 9.

**Proof.** If \( n = a_k a_{k-1} \ldots a_1 a_0 \), then \( n - S(n) = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10 a_1 + a_0 - a_k - a_{k-1} - \cdots - a_1 - a_0 = (10^k - 1) a_k + (10^{k-1} - 1) a_{k-1} + \cdots + 9 a_1 \). But \( 10^i - 1 = 99 \ldots 9 \) is obviously divisible by 9. \( \square \)

**Lemma 2.** For every positive integer \( n \) we have \( S(n) \leq 9(\lceil \log n \rceil + 1) \), where \( \lceil x \rceil \) denotes the largest integer not greater than \( x \), and \( \log x \) is the decimal logarithm.

**Proof.** If \( n = a_k a_{k-1} \ldots a_1 \) has \( k \) digits, then \( 10^{k-1} \leq n < 10^k \), so \( k - 1 \leq \log n < k \), i.e., \( \lceil \log n \rceil = k - 1 \). We also have \( S(n) = a_k + a_{k-1} + \cdots + a_1 \leq 9 k = 9(\lceil \log n \rceil + 1) \). \( \square \)

**Problem 1.**

By Lemma 2, we have \( S(5^n) \leq 9(\lceil \log 5^n \rceil + 1) = 9(\lceil n \log 5 \rceil + 1) < 9(0.7n + 1) \). So, we have \( 2^n \leq 6.3n + 9 \), which implies \( n \leq 5 \). It follows that we have to check \( n = 1, 2, 3, 4, 5 \). The corresponding values of \( 5^n \) are 5, 25, 125, 625, 3125. Their sums of digits are 5, 7, 8, 13, 11. So, the answer is \( n = 3 \).

**Problem 2.**

Using Lemma 2, we get \( S(2018^{2018}) \leq 9(\lceil 2018 \cdot \log 2018 \rceil + 1) = 9 \cdot 6670 = 60030 \). If \( n \leq 60030 \), then \( S(n) \leq S(59999) = 41 \). Consequently, \( S(S(2018^{2018})) \leq 41 \), therefore \( S(S(S(2018^{2018}))) \leq S(39) = 12 \).

The residue of division of 2018 by 9 is 2. The residues of \( 2018^m \) modulo 9 are therefore the same as for \( 2^m \), which are (for \( m = 0, 1, 2, \ldots \)): 1, 2, 4, 8, 7, 5, 1, 2, 4, 8, 7, 5, \ldots. This sequence is periodic with period 6. Since the residue of 2018 modulo 6 is 2, we conclude that 2018\(^{2018} \) gives residue 4 modulo 9. Among the numbers 1, 2, \ldots, 12 only 4 will have this residue. Consequently, the answer is 4.

**Problem 3.**

We have \( n \leq 2017 \). It follows that \( S(n) \leq S(1999) = 28 \). Hence, \( S(S(n)) \leq 10 \), and \( S(S(S(n))) \leq 9 \). Consequently, \( n = 2018 - S(n) - S(S(n)) - S(S(S(n))) \geq 2018 - 28 - 10 - 9 = 1971 \).

If \( r \) is the residue of \( n \) modulo 9, then the residue of \( n + S(n) + S(S(n)) + S(S(S(n))) \) is \( 4r \). Note that \( 4r \) is congruent to 2 modulo 9 if and only
if $r$ is congruent to 5 modulo 9. The residue of 2018 is 2, which implies that $r = 5$. It follows that the only candidates for $n$ are
The corresponding values of $n + S(n) + S(S(n)) + S(S(S(n)))$ are
\[
\begin{align*}
1976 + 23 + 5 + 5 &= 2009 \\
1985 + 23 + 5 + 5 &= 2018 \\
1994 + 23 + 5 + 5 &= 2027 \\
2003 + 5 + 5 + 5 &= 2018 \\
2012 + 5 + 5 + 5 &= 2027.
\end{align*}
\]
The answer is $n = 1985$ and 2003.

**Problem 4.**

a) Let us prove the inequality by induction. It is obviously true if one of the numbers $m, n$ is equal to zero. Let $m, n$ be positive integers. Let $m_0$ and $n_0$ be the last digits of $m, n$, respectively. Then $m = 10m_1 + m_0$ and $n = 10n_1 + n_0$, where $m_1$ and $n_1$ are obtained from $m, n$ by erasing the last digits. We have $m_1 < m$ and $n_1 < n$, and we may assume, by the inductive hypothesis, that $S(m_1 + n_1) \leq S(m_1) + S(n_1)$. If $m_0 + n_0 < 10$, then the last digit of $m + n$ is $m_0 + n_0$, and we have
\[
S(m + n) = S(10(m_1 + n_1) + m_0 + n_0) = S(m_1 + n_1) + m_0 + n_0 \leq S(m) + S(n).
\]
If $m_0 + n_0 \geq 10$, then the last digit of $m + n$ is $m_0 + n_0 - 10$ and
\[
S(m + n) = S(10(m_1 + n_1 + 1) + m_0 + n_0 - 10) = S(m_1 + n_1 + 1) + m_0 + n_0 - 10 \leq S(m) + S(n) - 9 < S(m) + S(n).
\]

b) It follows directly from inequality (a) that
\[
S(mn) = S(n^{m \text{ times}}) \leq mS(n).
\]
Write $m = a_k a_{k-1} \ldots a_1 a_0$. Then, using (a) and the inequality $S(mn) \leq mS(n)$, we get
\[
S(mn) = S(10^k a_k n + 10^{k-1} a_{k-1} n + \ldots + 10 a_1 n + a_0 n) \leq S(10^k a_k n) + S(10^{k-1} a_{k-1} n) + \ldots + S(10 a_1 n) + S(a_0 n) = a_k S(n) + a_{k-1} S(n) + \ldots + a_0 S(n) = (a_k + a_{k-1} + \ldots + a_0) S(n) = S(m) S(n).
\]

**Problem 5.**
Use Problem 4:

\[ S(n) = S(1000 \cdot n) = S(125 \cdot 8n) \leq S(125)S(8n) = 8S(8n) \]

and

\[ S(n) = S(10^5n) = S(2^5 \cdot 5^5n) \leq S(32)S(5^5n) = 5S(5^5n). \]

**Problem 6.**

Let us assume at first that the last digit of \( x \) is not 0. We have \( x(10^n - 1) = 10^n x - x \). Write \( x = a_n a_{n-1} \ldots a_1 \) where we allow leading digits (e.g., \( a_n \)) to be equal to 0. Then

\[ x(10^n - 1) = 10^n x - x = a_n a_{n-1} \ldots a_1 \underbrace{00 \ldots 0}_n - a_n a_{n-1} \ldots a_1 = a_n a_{n-1} \ldots a_2 (a_1 - 1)(9 - a_n)(9 - a_{n-1}) \ldots (9 - a_2)(10 - a_1), \]

and \( S(x(10^n - 1)) = a_n + a_{n-1} + \ldots + a_2 + a_1 - 1 + 9 - a_n + 9 - a_{n-1} + \ldots + 9 - a_2 + 10 - a_1 = 9n. \)

Suppose that \( x \) ends with \( k \) zeros. Then \( x = 10^k y \), where the last digit of \( y \) is not zero. We have \( 1 \leq y < 10^n \), hence, by the proven above, we have \( S(y(10^n - 1)) = 9n. \) But \( S(x(10^n - 1)) = S(y(10^n - 1)) \).

**Problem 7.**

We have \( 9 \cdot 99 \cdot 9999 \cdot \ldots \cdot 999 \ldots 99 = (10 - 1)(10^2 - 1)(10^4 - 1) \ldots (10^{2^n - 1} - 1) \).

Let \( x = (10 - 1)(10^2 - 1) \ldots (10^{2^n - 1} - 1) \). We have then

\[ x < 10 \cdot 10^2 \cdot 10^4 \ldots 10^{2^n-1} = 10^{1+2+2^2+\ldots+2^n-1} = 10^{2^n-1} < 10^{2^n}. \]

It follows from the previous problem that the answer is \( 9 \cdot 2^n \).

**Problem 8.**

Denote \( f(x) = x + S(x) \). If \( x \) does not end with 9, then \( f(x + 1) = f(x) + 2 \). If \( x \) ends with exactly \( k \) nines, then all of them will become zeros in \( x + 1 \), and the last non-nine digit will increase by 1. It follows that in this case \( f(x + 1) = f(x) - 9k + 2 \).

Let \( x \) be the largest positive integer such that \( f(x) \leq n \) (it exists, since we always have \( f(x) > x \)). Then \( f(x + 1) = f(x) + 2 \), since otherwise \( f(x + 1) \leq f(x) - 7 < n \), which contradicts the choice of \( x \).

We have either \( f(x) = n \), or \( f(x) \leq n - 1 \). If \( f(x) \leq n - 2 \), then we get \( f(x + 1) = f(x) + 2 \leq n \), which is a contradiction. Consequently, either \( f(x) = n \) or \( f(x + 1) = n + 1 \).

**Problem 9.** We will prove that there exist arbitrarily large sets of numbers with the same value of \( x + S(x) \). We will construct such sets by showing inductively how to construct sets \( A_n \) consisting of \( 2^n \) numbers with the same value of \( x + S(x) \).
Let \( f(x) = x + S(x) \), as in the previous problem. Suppose that \( 10^N > n \) and consider
\[
f(9 \cdot 10^N + n) = 9 \cdot 10^N + n + 9 + S(n) = 9 \cdot 10^N + 9 + f(n),
\]
and
\[
f(9 \cdot 10^N - n) = 9 \cdot 10^N - n + 8 + 9N + 1 - S(n) = 9 \cdot 10^N + 9 + 9N - f(n).
\]
The obtained two expressions are equal if and only if \( f(n) = 9N - f(n) \), i.e., if \( N = \frac{2f(n)}{9} \). We can find such an \( N \) if \( f(n) \) is divisible by 9. Note that then \( f(9 \cdot 10^N + n) = 9 \cdot 10^N + 9 + f(n) \) and \( f(9 \cdot 10^N - n) = 9 \cdot 10^N + 9 + 9N - f(n) \) are also divisible by 9.

We can use now this observation to construct the sets \( A_n \) inductively. Each set \( A_n \) will have \( 2^n \) numbers with the same value of \( f(x) \) such that \( f(x) \) is divisible by 9. Let us start with \( A_0 = \{9\} \). If \( A_n \) is constructed, and \( f(x) = a \) for all \( x \in A_n \), where \( a \) is divisible by 9, then the set \( A_{n+1} \) consists of numbers \( 9 \cdot 10^{2a} + x \) and \( 9 \cdot 10^{2a} - x \) for all \( x \in A_n \). Then it follows from the previous paragraph that for all elements \( y \in A_{n+1} \) the value of \( f(y) \) is the same and divisible by 9.

Let us show that the size of \( A_{n+1} \) is be twice the size of \( A_n \). It is enough to show that on each step we have that \( 10^{2a} \) is bigger than any element of \( A_n \). But \( a = f(x) > x \) for every \( x \in A_n \), and \( 10^{2a} > a \) for every \( a \geq 1 \).

Problem 10.

Let \( 0 \leq k_1 < k_2 < \ldots < k_{1000} \), and consider the number \( n = 10^{k_{1000}} + 10^{k_{999}} + \cdots + 10^{k_1} \). We have \( S(n) = 1000 \). We have \( n^2 = 10^{2k_{1000}} + 10^{2k_{999}} + \cdots + 10^{2k_1} + 2\sum_{1 \leq i_1 < i_2 \leq 1000} 10^{k_{i_1} + k_{i_2}} \). If all the numbers \( k_{i_1} + k_{i_2} \) for \( 1 \leq i_1 \leq i_2 \leq 1000 \) are pairwise different, then \( S(n^2) = 1000 + 2 \cdot \frac{1000 \cdot 999}{2} = 1000^2 \). The numbers \( k_{i_1} + k_{i_2} \) are pairwise different if the sequence \( k_i \) grows sufficiently fast. For instance, we can take \( k_i = 2^i \).

Problem 11. Squares of integers give only residues 0, 1, 4, and 7 modulo 9. Consequently, their sums of digits may give only these residues. In particular, 2018 can not be equal to \( S(n^2) \), as it is congruent to 2 modulo 9. On the other hand, we have
\[
\begin{align*}
S((10^k - 1)^2) &= S(10^{2k} - 2 \cdot 10^k + 1) = S(\underbrace{99 \ldots 98}_{k} \underbrace{00 \ldots 01}_{k}) = 9k, \\
S((10^k - 2)^2) &= S(10^{2k} - 4 \cdot 10^k + 4) = S(\underbrace{99 \ldots 96}_{k} \underbrace{00 \ldots 04}_{k}) = 9k + 1, \\
S((10^k - 3)^2) &= S(10^{2k} - 6 \cdot 10^k + 9) = S(\underbrace{99 \ldots 94}_{k} \underbrace{00 \ldots 09}_{k}) = 9k + 4, \\
S((10^{k+1} - 5)^2) &= S(10^{2k+2} - 10^{k+2} + 25) = S(\underbrace{99 \ldots 90}_{k} \underbrace{00 \ldots 025}_{k+2}) = 9k + 7,
\end{align*}
\]
which implies that all positive integers congruent to 0, 1, 4, or 7 modulo 9 are sums of digits of squares. In particular, there exists \( n \) such that \( S(n^2) = 2017 \).