Solutions 2019 AB Exam

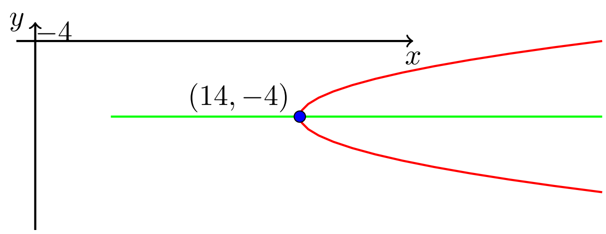
Texas A&M High School Math Contest
November 9, 2019

All answers must be simplified, and if units are involved, be sure to include them.

1. What is the equation of the axis of symmetry for the graph of $x = y^2 + 8y + 30$?

   ANSWER: $y = -4$

   Solution: $x = y^2 + 8y + 30 = (y + 4)^2 + 14$
   The graph is a parabola with vertex $(14, -4)$ opening to the right



2. A positive integer n written in base b is 25_b. If $2n$ is 52_b, what is b?

   ANSWER: 8

   Solution: $n = 2b + 5$
   $$2n = 4b + 10 = 5b + 2$$
   $$b = 8$$

3. Let n be a positive integer and let $S = 1 + 2 + 3 + \cdots + 10^n$. How many factors of 2 are there in the prime factorization of S?

   ANSWER: $n - 1$

   Solution:

$$S = \frac{(10^n)(10^n + 1)}{2} = \frac{2^n 5^n (10^n + 1)}{2}$$
$$= 2^{n-1} 5^n (10^n + 1)$$

   where $10^n + 1$ is odd.

4. (Multiple choice) In the real number system the polynomial $x^4 + 4$

   (A) does not factor into two polynomials of smaller degree.

(B) factors into a cubic and a linear polynomial.

(C) factors into two quadratics.

(D) factors into four linear polynomials.

(E) factors into a quadratic and two linear polynomials.

ANSWER: C

Solution: $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

Discussion:

$x^4 + 4$ clearly has no real roots so it has no linear factors. This eliminates B,D, and E.

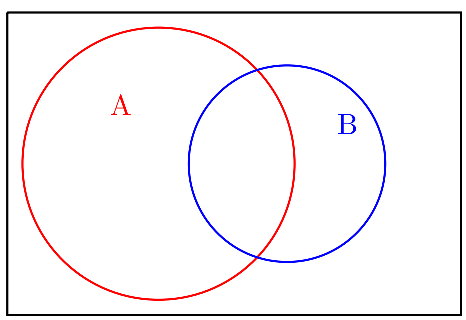
$x^4 + 4$ has four roots, so all roots are complex. Complex roots occur in conjugate pairs $c, \overline{c}, d, \overline{d}$.

$x^4 + 4 = (x - c)(x - \overline{c})(x - d)(x - \overline{d})$ where $(x - c)(x - \overline{c})$ and $(x - d)(x - \overline{d})$ have real coefficients. So C is true.

5. On a math exam every question gives two explicit choices, A and B, but the answer can be any combination of these two choices (*A and not B*, *not A and not B*, etc.) The answer includes choice A 70% of the time and choice B 50% of the time. The answer *neither A nor B* occurs 10% of the time. For what percentage of questions is the answer *both A and B*?

ANSWER: 30%

Solution:



$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$.90 = .70 + .50 - |A \cap B|$$
$$|A \cap B| = 1.20 - .90 = .30$$

6. The numerals for the counting numbers are written in (increasing) order beginning with 1. What is the 125th digit written?

ANSWER: 7

Solution:

$$\underset{9}{\underline{1...9}}\quad \underset{20}{\underline{10\,11...19}}\quad \underset{20}{\underline{20...29}}\quad \underset{20}{\underline{30...39}}\quad \underset{20}{\underline{40...49}}\quad \underset{20}{\underline{50...59}}\quad \underset{20}{\underline{60...69}}$$

$125 - 9 = 116$
$116 = 5(20) + 16$

$$\underset{16}{\underline{\underbrace{60\,61\,62\,63\,64\,65\,66\,67}}}\,68\,69$$

7. The correct formula for converting a Celsius temperature (C) to a Fahrenheit temperature (F) is given by $F = \frac{9}{5}C + 32$. To approximate the Fahrenheit temperature, Hasse doubles C and then adds 30. What is the largest magnitude (absolute value) of the error in this approximation for $-20 \le C \le 35$?

ANSWER: 6

Solution: We have

$$F = \tfrac{9}{5}C + 32$$
$$f = 2C + 30,$$

where f is the Hasse approximation. Then the absolute value of the error is

$$|F - f| = |\tfrac{18}{10}C - \tfrac{20}{10}C + 2|$$
$$= |-\tfrac{1}{5}C + 2|$$

The largest value occurs when $C = 35$ or $C = -20$

$$C = -20 \qquad |F - f| = 6$$

$$C = 35 \qquad |F - f| = 5$$

8. Find all values of x such that

$$\frac{x^2 + x + 4}{2x + 1} = \frac{4}{x}.$$

ANSWER: $-2, -1, 2$

Solution:

$$x^3 + x^2 + 4x = 8x + 4$$

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= 0 \\ x^2(x+1) - 4(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} (x^2 - 4)(x+1) &= 0 \\ (x-2)(x+2)(x+1) &= 0 \end{aligned}$$

$$x = 2, -2, -1$$

9. Given a quadratic polynomial $p(x) = x^2 + bx + c$ such that $p(2) = 0$. Find the value of $p(-1) + p(5)$.

ANSWER: 18

Solution: $p(x) = (x-2)(x-a)$ for some second root a.

$$\begin{aligned} p(-1) &= -3(-1-a) = 3 + 3a \\ p(5) &= 3(5-a) = 15 - 3a \end{aligned}$$

$$p(-1) + p(5) = 18$$

10. A bag contains 40 balls each of which is either black or gold. Mike reaches into the bag and randomly removes two balls. Each ball in the bag is equally likely to be removed. If the probability that two gold balls are removed is $\frac{5}{12}$, how many of the 40 balls are gold?

ANSWER: 26

Solution: Let G be the number of gold balls. Then

$$\frac{G}{40} \cdot \frac{G-1}{39} = \frac{5}{12}$$

$$\begin{aligned} 12(G^2 - G) &= 5 \cdot 40 \cdot 39 \\ G^2 - G &= 5 \cdot 10 \cdot 13 = 650 \\ G^2 - G - 650 &= 0 \\ (G+25)(G-26) &= 0 \\ G &= 26 \end{aligned}$$

11. A 200 gram solution consists of water and salt. 25% of the total mass of the solution is salt. How many grams of water needs to be added in order to change the solution so that it is 10% salt by mass?

ANSWER: 300 (grams)

Solution: The amount of salt is $\frac{1}{4}(200) = 50$g. The amount of water initially is therefore $200 - 50 = 150$g.

Let x be the amount of water added. The new solution is $(150 + x) + 50 = 200 + x$, with 50g of salt. We want x so that

$$\frac{50}{200 + x} = \frac{1}{10} \implies 500 = 200 + x \implies x = 300 \text{ g}$$

12. What is the largest two-digit integer that becomes 75% greater when its digits are reversed?

ANSWER: 48

Solution: Let $x = 10a + b$. Reversing the digits gives $\overline{x} = 10b + a$. We want a, b such that

$$\overline{x} = x + \tfrac{3}{4}x = \tfrac{7}{4}x$$

$$\begin{aligned} 10b + a &= \tfrac{7}{4}(10a + b) \\ 40b + 4a &= 70a + 7b \\ 33b &= 66a \\ b &= 2a \end{aligned}$$

The largest possible value of b is 8; hence the largest possible value of a is 4. Therefore $x = 48$.

13. The equation $4x^2 - y^2 = 11$ has exactly one integer pair solution (x, y) with both $x > 0$ and $y > 0$. Find it.

ANSWER: $(3, 5)$

Solution:

$$\begin{aligned} 4x^2 - y^2 &= 11 \\ (2x - y)(2x + y) &= 11 \end{aligned}$$

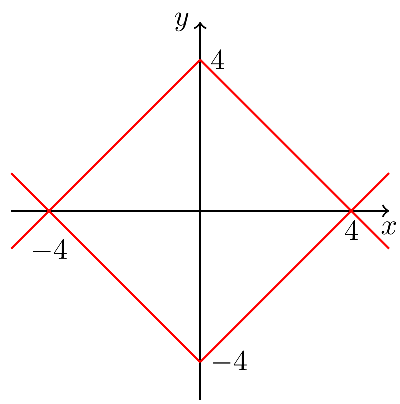
Since x and y are positive integers the terms $2x-y$ and $2x+y$ must equal 1 and 11 respectively. Thus,

$$\begin{aligned} 2x + y &= 11 \\ 2x - y &= 1 \implies 4x = 12 \implies x = 3 \text{ and } y = 5 \end{aligned}$$

14. Compute the area of the region bounded by the graphs of $y = 4 - |x|$ and $y = |x| - 4$.

ANSWER: 32

Solution:



The region is a square, which consists of two congruent triangles. Each of which has area equal to $\frac{4 \cdot 8}{2} = 16$. Thus, the enclosed area equals 32.

15. Determine all ordered pairs (a, b) of real numbers that satisfy the following system of equations

$$a + b = 16,$$

$$\frac{1}{a} + \frac{1}{b} = \frac{4}{7}.$$

ANSWER: $(2, 14)$ and $(14, 2)$

Solution:

$$a + b = 16$$

$$\frac{b + a}{ab} = \frac{4}{7}$$

$$\frac{16}{ab} = \frac{4}{7}$$
$$ab = 28$$
$$b = \frac{28}{a}$$

$$\begin{aligned}
&16 = a + b = a + \tfrac{28}{a} \\
&16a = a^2 + 28 \\
&a^2 - 16a + 28 = 0 \\
&(a-2)(a-14) = 0 \\
&a = 2 \quad , \quad b = 14 \\
&a = 14 \quad , \quad b = 2
\end{aligned}$$

16. The numbers a, b, c are in geometric progression in that order, and they are the lengths of the sides of a right triangle with c being the length of the hypotenuse. Find $\dfrac{b}{a}$.

ANSWER: $\dfrac{\sqrt{2+2\sqrt{5}}}{2}$

Solution: There is an $\alpha > 0$ such that $b = \alpha a$, $c = \alpha^2 a$. Note that $\frac{b}{a} = \alpha$.

$$\begin{aligned}
&a^2 + b^2 = c^2 \\
&a^2 + \alpha^2 a^2 = \alpha^4 a^2 \qquad (a \neq 0) \\
&1 + \alpha^2 = \alpha^4 \\
&\alpha^4 - \alpha^2 - 1 = 0
\end{aligned}$$

$\alpha^2 = \dfrac{1 \pm \sqrt{5}}{2}$. Since $\alpha^2 > 0$,

$\alpha^2 = \dfrac{1 + \sqrt{5}}{2}$. Since $\alpha > 0$

$$\alpha = \sqrt{\tfrac{1+\sqrt{5}}{2}} = \frac{\sqrt{1+\sqrt{5}}}{\sqrt{2}} = \frac{\sqrt{2+2\sqrt{5}}}{2} = \frac{b}{a}$$

17. If $\dfrac{a}{b}$ is the reduced fraction that equals $0.571717\ldots$, find $a + b$.

ANSWER: 778

Solution:

$$\begin{aligned}
x &= .571717... \\
100x &= 57.1717... \\
10000x &= 5717.1717... \\
10000x - 100x &= 5717 - 57 \\
9900x &= 5660
\end{aligned}$$

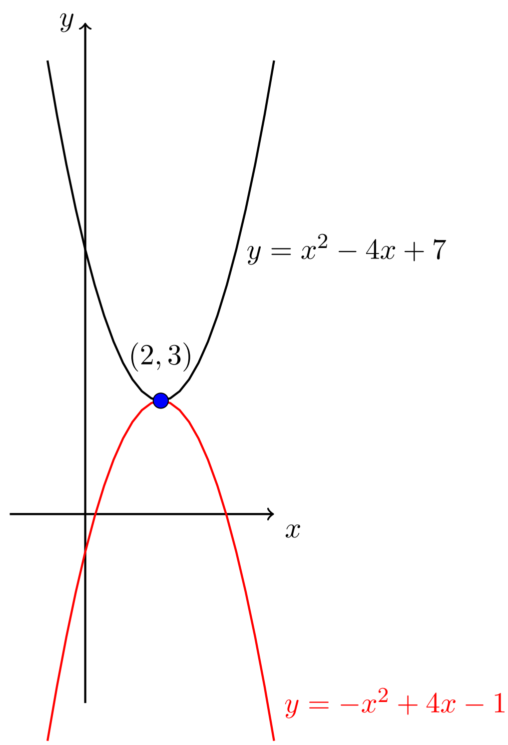
$$x = \frac{5660}{9900} = \frac{566}{990} = \frac{283}{495} = \frac{a}{b}$$

$$a + b = 283 + 495 = 778$$

18. The parabola $y = x^2 - 4x + 7$ is reflected about the line $y = 3$ to obtain another parabola. Find an equation for this parabola in the form $y = ax^2 + bx + c$.

    ANSWER: $y = -x^2 + 4x - 1$

    Solution:



$y = x^2 - 4x + 7 = (x-2)^2 + 3$. So the vertex of the first parabola is $(2, 3)$. This is also the vertex of the second parabola. It opens down, so

$$\begin{aligned} y &= -(x-2)^2 + 3 \\ &= -x^2 + 4x - 1 \end{aligned}$$

19. Daisy likes to paddle her raft down a river from point A to point B. The speed of the current in the river is always the same. Daisy always paddles the raft at the same constant speed. On days when she paddles with the current, it takes her 18 minutes to go from A to B. When she does not paddle the current carries her from A to B in 30 minutes. If there were no current, how long would it take her to paddle from A to B?

ANSWER: 45 minutes

Solution:

$$\begin{aligned} r &= \text{ rate of raft in still water} \\ v &= \text{ speed of river} \\ d &= \text{ distance from A to B} \end{aligned}$$

$$\begin{array}{c|c} d = (r+v)18 & d = rt \\ d = 30v & 30v = \frac{2}{3}vt \\ (r+v)18 = 30v & 30 = \frac{2}{3}t \\ 18r = 12v & t = \frac{90}{2} = 45 \\ 3r = 2v & \\ r = \frac{2}{3}v & \end{array}$$

20. Suppose L is a line that passes through the first quadrant. The positive x-axis, the positive y-axis and L form the boundary of a triangular shaped region whose area is 20. If L contains the point $(6, -5)$, find its y-intercept.

ANSWER: 10

Solution: L has an equation of the form $y = mx + b$ where b is the y-intercept. Let $(a, 0)$ be the x-intercept. Then

$$\begin{aligned} \tfrac{1}{2}ab &= 20 \\ ab &= 40 \end{aligned}$$

Also

$$m = -\tfrac{b}{a}$$

$$y = -\tfrac{b}{a}x + b$$

$$\begin{aligned} -5 &= -\tfrac{b}{a}6 + b \\ -5a &= -6b + ab = -6b + 40 \\ -5(\tfrac{40}{6}) &= -6b + 40 \\ -200 &= -6b^2 + 40b \end{aligned}$$

$$\begin{aligned} 6b^2 - 40b - 200 &= 0 \\ 3b^2 - 20b - 100 &= 0 \\ (3b + 10)(b - 10) &= 0 \end{aligned}$$

$$\underline{b = 10} \quad \text{because } b > 0.$$

2019 AB Exam Answers

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1. $y = -4$
2. 8
3. $n - 1$
4. C
5. 30%
6. 7
7. 6
8. $-2, -1, 2$
9. 18
10. 26
11. 300 (grams)
12. 48
13. $(3, 5)$
14. 32
15. $(2, 14)$ and $(14, 2)$
16. $\dfrac{\sqrt{2 + 2\sqrt{5}}}{2} \left(= \sqrt{\frac{1+\sqrt{5}}{2}} = \frac{\sqrt{1+\sqrt{5}}}{\sqrt{2}}\right)$
17. 778
18. $y = -x^2 + 4x - 1$
19. 45 (minutes)
20. 10