BC Exam
Texas A&M High School Math Contest
November 9, 2019

1. How many quadruples (a, b, c, d) of positive integers are there such that $a \le b \le c \le d$ and
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 2?$$

2. A 5×5 square drawn on the square grid is then cut into smaller squares (all cuts go along the grid lines). What is the minimal possible number of pieces in such a partition?

3. For any positive integer n let $S(n)$ denote the sum of its digits (in decimal notation). Find all integers n such that $n + S(n) = 2019$.

4. Find the area of the region bounded by the curves $y = |x - 2| - 1$ and $y = 3 - |x|$.

5. Evaluate the product $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{2019^2}\right)$.

6. Let ABC be an acute triangle. Let AH_1, BH_2 and CH_3 be the altitudes of this triangle. Find the length of the side AB if $|AH_1| = |BH_2| = 12$ and $|CH_3| = 10$.

7. The equation $x^2 - x - 5 + \sqrt{x^2 - x + 1} = 0$ has two real solutions. Find their sum.

8. Find a positive integer n such that $1^2 + 2^2 + \cdots + n^2 = 70^2$. [Hint: the formula for the sum is $\frac{1}{6}n(n+1)(2n+1)$.]

9. A parallelogram is inscribed in a circle of radius 2 and circumscribed about a circle of radius $\sqrt{2}$. Find the length of the shortest side of the parallelogram.

10. Find all pairs (x, y) satisfying the system
$$\begin{cases} x^3 + y^3 = xy(x + y), \\ x^2 + y^2 = 8. \end{cases}$$

11. The interior angles of a certain convex polygon add up to 900 degrees. How many sides does the polygon have?

12. Find all integers n in the range from 50 to 100 such that the fraction $\dfrac{3n + 2}{13n - 1}$ is not reduced to lowest terms.

13. Let ABC be an obtuse triangle. Let AD be the altitude and AE be the angle bisector of the triangle ABC. Find the length of the side BC if $|BE| = |DE| = 3$ and $|AE| = 6$.

14. Find the smallest positive integer that has exactly 2019 different divisors. [Hint: the number is too big to write out; find some expression for it.]

15. A regular dodecagon (12-gon) is inscribed into a circle of radius 1. How many diagonals of the dodecagon intersect the concentric circle of radius 2/3?

16. Eight coins are arranged in a row and numbered from left to right (the leftmost is the first, the rightmost is the eighth). We start turning the coins over, one coin at a time, according to the following rule: if we see k heads (and $8-k$ tails) then the k-th coin is turned over next. We keep turning the coins over until we see eight tails and no heads. Then we are done. What is the maximal possible number of turns?

17. Three solid balls of radius 1 are placed on a horizontal floor so that they touch one another. The balls are firmly attached to the floor and cannot move. The fourth ball of radius 1 is put into a hole formed by the first three balls. Find the clearance between the fourth ball and the floor.

18. Find all integers n between 100 and 200 such that the number $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ (n factorial) is divisible by 2^{n-1}.

19. A right triangle with sides of length 3, 4 and 5 is cut out of paper. One folds the triangle along a straight line so that the folded figure is also a triangle. What is the minimal possible area of the new triangle?

20. An integer-valued function $f(n)$ of an integer argument n satisfies a functional equation $f(f(x)) + 2f(y) = f(x+2y) - 3$ for all integers x and y. Find $f(5)$.