BC Exam (with solutions)
Texas A&M High School Math Contest
November 9, 2019

1. How many quadruples (a, b, c, d) of positive integers are there such that $a \le b \le c \le d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 2?$$

Answer: 4. [The quadruples are $(1, 2, 3, 6)$, $(1, 2, 4, 4)$, $(1, 3, 3, 3)$ and $(2, 2, 2, 2)$.]

If $a > 2$ then $a^{-1} + b^{-1} + c^{-1} + d^{-1} \le 4a^{-1} < 2$. Hence a is at most 2. If $a = 2$ then $a^{-1} + b^{-1} + c^{-1} + d^{-1} \le 4a^{-1} = 2$. Moreover, the equality is attained only if $a = b = c = d = 2$. This gives us one quadruple $(2, 2, 2, 2)$. To find the others, we need to set $a = 1$.

If $b = 1$ then $a^{-1} + b^{-1} + c^{-1} + d^{-1} > a^{-1} + b^{-1} = 2$. Hence $b \ge 2$. If $b \ge 3$ then $a^{-1} + b^{-1} + c^{-1} + d^{-1} \le 1 + 3b^{-1} \le 2$. Moreover, the equality is attained only if $b = c = d = 3$. We find another quadruple $(1, 3, 3, 3)$ and set $b = 2$ for the remainder of our search.

Now $a = 1$ and $b = 2$ so that $c^{-1} + d^{-1} = 1/2$. Since $c^{-1} < c^{-1} + d^{-1} \le 2c^{-1}$, it follows that $2 < c \le 4$. Hence $c = 3$ or 4. Then $d = 6$ or 4, respectively. Thus the remaining quadruples are $(1, 2, 3, 6)$ and $(1, 2, 4, 4)$.

2. A 5×5 square drawn on the square grid is then cut into smaller squares (all cuts go along the grid lines). What is the minimal possible number of pieces in such a partition?

Answer: 8.

First we describe a partition into 8 pieces. It begins with two cuts, one horizontal and one vertical, each dividing the 5×5 square in proportion $2 : 3$. The cuts produce one 3×3 square, one 2×2 square, and two rectangles of dimensions 2×3 and 3×2. Then each rectangle is further cut into one 2×2 square and two 1×1 squares. Overall we have one 3×3 square, three 2×2 squares and four 1×1 squares.

Now we are going to show that any partition has at least 8 pieces. First assume we have a 4×4 piece. Then the rest must be cut into 1×1 squares. Since the area of the entire square is $5^2 = 25$ (where the unit area is that of a 1×1 square) and the area of the large piece is $4^2 = 16$, we will have 9 unit squares, for a total of 10 pieces.

Next assume we have a 3×3 piece. Such a piece contains the central grid square, hence it is unique. Let us slice the 5×5 square into five horizontal strips of dimensions 1×5. If a $k \times k$ square intersects a strip, there are exactly k unit squares in the intersection. It follows that every strip intersects a piece of odd dimensions. Since the 3×3 square intersects only three strips, we must have at least two 1×1 pieces. From area considerations, $5^2 = 3^2 + 2^2 k_2 + k_1$, where k_1 is the number of 1×1 pieces and k_2 is the number of 2×2 pieces. Then $k_2 = (16 - k_1)/4 \le (16 - 2)/4 = 3.5$ so that $k_2 \le 3$. Also, $k_1 = 16 - 4k_2$. Hence the total number of pieces is $1 + k_2 + k_1 = 17 - 3k_2 \ge 17 - 3 \cdot 3 = 8$.

It remains to consider the case when we have only 1×1 and 2×2 pieces. Let k_1 be the number of 1×1 pieces and k_2 be the number of 2×2 pieces. From area considerations, $k_1 + 4k_2 = 25$. Slicing the 5×5 square into 1×5 strips as above, we obtain that each strip contains a 1×1 piece. Hence $k_1 \ge 5$. In fact, some strip contains more than one 1×1 piece so that $k_1 \ge 6$. Indeed, if the top and the bottom strips contain only one piece of dimensions 1×1, then there are two 2×2 pieces adjacent to the top side of the big square and another two 2×2 pieces adjacent to the bottom side. Between those four, there is no place for another 2×2 piece. Hence the middle strip is filled by 1×1 pieces. The inequality $k_1 \geq 6$ implies that $k_2 = (25 - k_1)/4 \leq (25 - 6)/4 = 4.75$ so that $k_2 \leq 4$. Then the total number of pieces is $k_1 + k_2 = (25 - 4k_2) + k_2 = 25 - 3k_2 \geq 25 - 3 \cdot 4 = 13$.

3. For any positive integer n let $S(n)$ denote the sum of its digits (in decimal notation). Find all integers n such that $n + S(n) = 2019$.

Answer: 1995, 2013.

Since $n \leq 2019$, the number n has at most four digits. Moreover, if n has exactly four digits, the first digit is at most 2. Hence $S(n) \leq 2 + 3 \cdot 9 = 29$. Then $n = 2019 - S(n) \geq 2019 - 29 = 1990$. It follows that the decimal representation of n is either $20d_1d_0$ or $199d_0$, where d_0 and d_1 are some decimal digits. In the first case, $n = 2000 + 10d_1 + d_0$ and $S(n) = 2 + d_1 + d_0$. Then $n + S(n) = 2002 + 11d_1 + 2d_0$. The latter equals 2019 if and only if $11d_1 + 2d_0 = 17$. This is possible only if $d_1 = 1$ and $d_0 = 3$.

In the second case, $n = 1990 + d_0$ and $S(n) = 19 + d_0$. Then the equation $n + S(n) = 2019$ is reduced to $2009 + 2d_0 = 2019$. The only solution is $d_0 = 5$.

4. Find the area of the region bounded by the curves $y = |x - 2| - 1$ and $y = 3 - |x|$.

Answer: 6.

The curve $y = |x - 2| - 1$ is a right angle, with the vertex $A = (2, -1)$ and opening up (that is, the angle bisector goes in the direction of the y-axis). The curve $y = 3 - |x|$ is also a right angle, with the vertex $B = (0, 3)$ and opening down (that is, the angle bisector goes in the direction opposite to that of the y-axis). The point B lies above the curve $y = |x - 2| - 1$ (since $3 > |0 - 2| - 1$), which means it is inside the right angle. Similarly, the point A lies inside the other right angle. It follows that the region R bounded by the two curves is a rectangle. The points A and B are opposite vertices of R. The other two vertices are points of intersection of the curves. To find them, we solve the system

$$\begin{cases} y = |x - 2| - 1, \\ y = 3 - |x|. \end{cases}$$

The system implies that $|x - 2| - 1 = 3 - |x|$, hence $|x| + |x - 2| = 4$. To solve the latter equation, we consider three cases: $x < 0$, $0 \leq x < 2$, and $x \geq 2$. In the first case the equation becomes $-x - (x - 2) = 4$, then $-2x = 2$ and $x = -1$ (note that $-1 < 0$). In the second case, we have $-x + (x - 2) = 4$, which yields no solution. In the third case, we obtain $x + (x - 2) = 4$, then $2x = 6$ and $x = 3$ (note that $3 \geq 2$). If $x = -1$ then $y = 2$. If $x = 3$ then $y = 0$.

Let $C = (3, 0)$. By the above AC and BC are adjacent sides of the rectangle R. We obtain that $|AC| = \sqrt{(3 - 2)^2 + (0 - (-1))^2} = \sqrt{2}$ and $|BC| = \sqrt{(3 - 0)^2 + (0 - 3)^2} = 3\sqrt{2}$. Then the area of R equals $|AC| \cdot |BC| = \sqrt{2} \cdot 3\sqrt{2} = 6$.

5. Evaluate the product $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\left(1 - \dfrac{1}{4^2}\right) \dots \left(1 - \dfrac{1}{2019^2}\right)$.

Answer: $\dfrac{1010}{2019}$.

Each factor in the product is of the form $1 - 1/n^2$, where n runs from 2 to 2019. We transform it as follows:

$$1 - \frac{1}{n^2} = \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right) = \frac{n - 1}{n} \cdot \frac{n + 1}{n}.$$

Now the product decomposes into two telescopic products:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2019^2}\right)=\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{2018}{2019}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\cdots\cdot\frac{2020}{2019}\right)$$
$$=\frac{1}{2019}\cdot\frac{2020}{2}=\frac{1010}{2019}.$$

6. Let ABC be an acute triangle. Let AH_1, BH_2 and CH_3 be the altitudes of this triangle. Find the length of the side AB if $|AH_1|=|BH_2|=12$ and $|CH_3|=10$.

Answer: 15.

Let $x=|AB|$. Since each of the products $|BC|\cdot|AH_1|$, $|AC|\cdot|BH_2|$ and $|AB|\cdot|CH_3|$ equals twice the area of the triangle ABC, it follows that $|AC|=|BC|=10x/12=5x/6$. In particular, the triangle ABC is isosceles. Hence the altitude CH_3 is also the median so that $|AH_3|=|H_3B|=x/2$. By the Pythagorean Theorem, $|AC|^2=|AH_3|^2+|CH_3|^2$. That is, $(5x/6)^2=(x/2)^2+10^2$. Then $10^2=(5x/6)^2-(x/2)^2=x^2(25/36-1/4)=4x^2/9$. Therefore $x^2=9\cdot 10^2/4=15^2$. Thus $x=15$.

7. The equation $x^2-x-5+\sqrt{x^2-x+1}=0$ has two real solutions. Find their sum.

Answer: 1.

Let $y=\sqrt{x^2-x+1}$. Then $x^2-x-5=(x^2-x+1)-6=y^2-6$ and the equation becomes $y^2+y-6=0$. This quadratic equation in y has roots 2 and -3. By construction, $y\geq 0$. Hence $y=2$. Returning to the original variable, we obtain $x^2-x+1=2^2$ or, equivalently, $x^2-x-3=0$. The latter equation has roots $\frac{1}{2}(1-\sqrt{13})$ and $\frac{1}{2}(1+\sqrt{13})$. Their sum equals 1.

Alternative solution: Suppose x_1 is a real solution of the equation. Let $x_2=1-x_1$. Then $x_2^2-x_2=x_2(x_2-1)=(1-x_1)(1-x_1-1)=x_1(x_1-1)=x_1^2-x_1$, which implies that x_2 is also a solution. Note that $x_2\neq x_1$ (otherwise $x_1=1/2$, but $1/2$ is not a solution). Hence x_1 and x_2 are the two real solutions of the equation. Their sum is $x_1+x_2=x_1+(1-x_1)=1$.

8. Find a positive integer n such that $1^2+2^2+\cdots+n^2=70^2$. [Hint: the formula for the sum is $\frac{1}{6}n(n+1)(2n+1)$.]

Answer: 24.

Since $1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$, we need to find an integer solution of the equation $n(n+1)(2n+1)=6\cdot 70^2$. Notice that any two of the numbers n, $n+1$ and $2n+1$ have no common divisors other than 1. Indeed, any common divisor of n and $n+1$ will also divide their difference $(n+1)-n=1$, any common divisor of n and $2n+1$ will also divide $(2n+1)-2n=1$, and any common divisor of $n+1$ and $2n+1$ will also divide $2(n+1)-(2n+1)=1$. Since the prime factorization of $6\cdot 70^2$ is $2^3\cdot 3\cdot 5^2\cdot 7^2$, it follows that one of the numbers n, $n+1$ and $2n+1$ is divisible by $7^2=49$. Clearly, $n<70$. Then $n+1<71<2\cdot 49$ and $2n+1<141<3\cdot 49$. Also, $2n+1\neq 2\cdot 49$ as it is odd. We conclude that one of the numbers n, $n+1$ and $2n+1$ is exactly 49. A quick check shows $2n+1$ is that number. Then $n=24=2^3\cdot 3$ and $n+1=25=5^2$.

9. A parallelogram is inscribed in a circle of radius 2 and circumscribed about a circle of radius $\sqrt{2}$. Find the length of the shortest side of the parallelogram.

Answer: $2\sqrt{2}$.

A quadrilateral can be inscribed in a circle if and only if the sum of opposite angles equals π. In any parallelogram, the opposite angles are equal. Therefore a parallelogram can be inscribed in a circle if and only if it is a rectangle.

A convex quadrilateral can be circumscribed about a circle if and only if the sums of lengths of opposite sides are equal. In any parallelogram, the opposite sides are of the same length. Therefore a parallelogram can be circumscribed about a circle if and only if it is a rhombus.

Thus our parallelogram is both a rectangle and a rhombus. Hence it is a square. Then all sides are of the same length, which is also equal to the diameter of the inscribed circle, $2\sqrt{2}$.

10. Find all pairs (x, y) satisfying the system

$$\begin{cases} x^3 + y^3 = xy(x + y), \\ x^2 + y^2 = 8. \end{cases}$$

Answer: $(-2, -2)$, $(-2, 2)$, $(2, -2)$, $(2, 2)$.

Since $x^3+y^3 = (x+y)(x^2-xy+y^2)$, the first equation is equivalent to $(x+y)(x^2-2xy+y^2) = 0$ or $(x + y)(x - y)^2 = 0$. Hence $x = y$ or $x = -y$. Substituting $x = \pm y$ into the second equation, we obtain $2y^2 = 8$, which has solutions $y = -2$ and $y = 2$. Then $x = -2$ or $x = 2$. Note that any combination of signs for x and y produces a solution of the system.

11. The interior angles of a certain convex polygon add up to 900 degrees. How many sides does the polygon have?

Answer: 7.

Let n denote the number of sides of the polygon. We choose a vertex and connect it to every other vertex by a segment. Two of these $n-1$ segments are sides of the polygon. Since the polygon is convex, the other $n - 3$ segments are diagonals. They cut the polygon into $n - 2$ triangles. Note that the sum of all (interior) angles of the polygon equals the sum of all angles of all triangles in the partition. Since the angles of any triangle add up to 180 degrees, there are $900/180 = 5$ triangles. Hence $n - 2 = 5$, then $n = 7$.

12. Find all integers n in the range from 50 to 100 such that the fraction $\dfrac{3n+2}{13n-1}$ is not reduced to lowest terms.

Answer: 67, 96.

For any integer n, the numbers $3n + 2$ and $13n - 1$ are different from zero. If the fraction $\dfrac{3n+2}{13n-1}$ is not reduced, then the fraction $\dfrac{13n-1}{3n+2}$ is not reduced as well. Since

$$\frac{13n-1}{3n+2} = 4 + \frac{n-9}{3n+2},$$

neither is the fraction $\dfrac{n-9}{3n+2}$. The same holds true for the fraction $\dfrac{3n+2}{n-9}$ (unless $n = 9$). Since

$$\frac{3n+2}{n-9} = 3 + \frac{29}{n-9},$$

the fraction $\dfrac{29}{n-9}$ is not reduced as well. The number 29 is prime. Hence $n-9$ is a multiple of 29 (this also includes the case $n=9$). Conversely, if $n-9=29k$ for some integer k, then $3n+2=3(9+29k)+2=29+3\cdot 29k=29(3k+1)$ and $13n-1=13(9+29k)-1=116+13\cdot 29k=29(13k+4)$ so that the fraction $\dfrac{3n+2}{13n-1}$ is not reduced.

Thus we are looking for numbers of the form $n=9+29k$, where k is an integer. There are two such numbers in the range from 50 to 100, namely, $67=9+2\cdot 29$ and $96=9+3\cdot 29$.

13. Let ABC be an obtuse triangle. Let AD be the altitude and AE be the angle bisector of the triangle ABC. Find the length of the side BC if $|BE|=|DE|=3$ and $|AE|=6$.

Answer: 5.

The points B, D and E lie on the same line BC. Since $|BE|=|DE|$, it follows that either E is the midpoint of the segment BD, or else D coincides with B. The latter would imply that the triangle ABC is right while it is assumed to be obtuse. Hence E is the midpoint of BD. As it will turn out below, $|BC|<|BD|$ so that the base D of the altitude AD lies on the extension of the side BC beyond the endpoint C.

Let $x=|EC|$. Then $|CD|=|3-x|$. The triangles ADE, ADB and ADC are right. By the Pythagorean Theorem, $|AD|^2=|AE|^2-|DE|^2=6^2-3^2=27$, then $|AB|^2=|BD|^2+|AD|^2=6^2+27=63$ and $|AC|^2=|AD|^2+|CD|^2=27+(3-x)^2=x^2-6x+36$.

The angle bisector AE divides the side BC of the triangle ABC into two segments whose lengths are proportional to lengths of the other two sides:

$$\frac{|AB|}{|BE|}=\frac{|AC|}{|EC|}.$$

Squaring both sides of this equality and substituting the lengths, we obtain

$$\frac{63}{9}=\frac{x^2-6x+36}{x^2},$$

which is simplified to $x^2+x-6=0$. The quadratic equation has two roots, 2 and -3. Only the positive root makes sense here. Thus $|EC|=2$, then $|BC|=|BE|+|EC|=3+2=5$.

14. Find the smallest positive integer that has exactly 2019 different divisors. [Hint: the number is too big to write out; find some expression for it.]

Answer: $2^{672}\cdot 3^2$.

Every integer $n\geq 2$ can be factored into a product of primes: $n=p_1^{d_1}p_2^{d_2}\dots p_k^{d_k}$, where $p_1,p_2,\dots,p_k$ are distinct primes and $d_1,d_2,\dots,d_k$ are positive integers. Then any divisor of n is of the form $p_1^{c_1}p_2^{c_2}\dots p_k^{c_k}$, where $0\leq c_i\leq d_i$, $i=1,2,\dots,k$. There are d_1+1 ways to choose c_1 (from 0 to d_1), d_2+1 ways to choose c_2, and so on. All choices are independent and in the end we get a unique divisor of n. Therefore the total number of divisors is $(d_1+1)(d_2+1)\dots(d_k+1)$.

Now we need to get a prime factorization of 2019. It is easy to see that 2019 is divisible by 3: $2019=3\cdot 673$. The number 673 turns out to be prime. To verify this, it is enough to check that 673 has no prime divisors in the range from 1 to $\sqrt{673}$. Note that $\sqrt{673}<\sqrt{729}=27$. Hence the prime numbers to check are $2,3,5,7,11,13,17,19$ and 23. None of them divides 673.

Since the prime factorization of 2019 is $3 \cdot 673$, it follows from the above that all numbers that have exactly 2019 divisors are of the form p^{2018}, where p is prime, or of the form $p_1^2 p_2^{672}$, where p_1 and p_2 are distinct primes. The smallest of them is, clearly, one of the numbers 2^{2018}, $2^2 \cdot 3^{672}$ and $3^2 \cdot 2^{672}$. To compare these numbers, observe that $2^3 < 3^2 < 2^4$. Then $2^{2018} > 3^2 \cdot 2^{2014} > 3^2 \cdot 2^{672}$ and $2^2 \cdot 3^{672} = 2^2 \cdot 3^2 \cdot (3^2)^{335} > 2^2 \cdot 3^2 \cdot (2^3)^{335} = 3^2 \cdot 2^{1007} > 3^2 \cdot 2^{672}$. Thus the smallest number is $3^2 \cdot 2^{672}$.

15. A regular dodecagon (12-gon) is inscribed into a circle of radius 1. How many diagonals of the dodecagon intersect the concentric circle of radius 2/3?

Answer: 30.

Let A and B be two vertices of the dodecagon and O be the center of the circle. If A and B are adjacent vertices, then AB is a side of the dodecagon and the angle $\angle AOB$ equals $2\pi/12 = \pi/6$. Otherwise AB is a diagonal and $\angle AOB = \pi k/6$, where $k = 2, 3, 4, 5$ or 6. In the case $k = 6$, the diagonal AB goes through the center O and hence intersects any concentric circle. There are 6 such diagonals. For $2 \le k \le 5$, there are 12 diagonals such that $\angle AOB = \pi k/6$. In the latter case, the points A, B and O are vertices of a triangle. Let OH be the altitude of this triangle. Since the triangle is isosceles, $|OA| = |OB| = 1$, the altitude OH is also the angle bisector. The diagonal AB intersects the concentric circle of radius 2/3 if the distance from the diagonal to the center, which equals $|OH|$, is less than 2/3. Since OHA is a right triangle, $\angle AHO = \pi/2$, we obtain that $|OH| = |OA| \cos\angle AOH = \cos\angle AOH = \cos\frac{1}{2}\angle AOB = \cos(\pi k/12)$.

Now we need to compare the numbers $\cos(\pi/6)$, $\cos(\pi/4)$, $\cos(\pi/3)$ and $\cos(5\pi/12)$ with 2/3. Recall that the cosine decreases as the angle increases. Therefore $\cos(5\pi/12) < \cos(\pi/3) = 1/2$, which is less than 2/3. Also, $\cos(\pi/6) > \cos(\pi/4) = \sqrt{1/2}$, which is greater than $2/3 = \sqrt{4/9}$. Thus the concentric circle of radius 2/3 is intersected by $6 + 12 + 12 = 30$ diagonals.

16. Eight coins are arranged in a row and numbered from left to right (the leftmost is the first, the rightmost is the eighth). We start turning the coins over, one coin at a time, according to the following rule: if we see k heads (and $8 - k$ tails) then the k-th coin is turned over next. We keep turning the coins over until we see eight tails and no heads. Then we are done. What is the maximal possible number of turns?

Answer: 36. [Achieved for a unique initial configuration: $TTTTHHHH$.]

Let us split all turns into batches of consecutive turns of the same kind. Suppose that the first turn is tails-to-heads. In this case, we first do $m_1 \ge 1$ tails-to-heads turns, then $m_2 \ge 1$ heads-to-tails turns, then $m_3 \ge 1$ tails-to-heads turns, and so on... In the case the first turn is heads-to-tails, we begin with $m_1 \ge 1$ heads-to-tails turns, which are followed by $m_2 \ge 1$ tails-to-heads turns, and so on...

The key observation is that $m_{i+1} > m_i$ for all i (whenever m_{i+1} is defined). Moreover, all coins turned over in the batch of m_i turns will be turned over again in the next batch. Indeed, let k_i be the number of the coin turned over first in the batch of m_i turns. First assume the turns in this batch are tails-to-heads. Then each turn increases the number of heads by 1 and so the next coin to be turned over is the first coin to the right of the previously turned one. Hence in this batch we turn over coins with consecutive numbers $k_i, k_i + 1, k_i + 2, \dots, k_i + m_i - 1$. After that we turn over the coin with number $k_i + m_i$, which is the first turn in the batch of m_{i+1} heads-to-tails turns. In this batch each turn decreases the number of heads by 1, hence we turn over coins with descending numbers $k_i + m_i, k_i + m_i - 1, \ldots$ The batch ends as soon as we reach a coin showing tails. By the above this cannot happen until we turn over all coins that were turned over in the previous batch. As a consequence, $m_{i+1} > m_i$. The argument is similar in the case when turns in the batch of m_i turns are heads-to-tails. In that case we first turn over coins with descending numbers $k_i, k_i - 1, \ldots, k_i - m_i + 1$. In the next batch we turn over coins with ascending numbers $k_i - m_i, k_i - m_i + 1, \ldots$ until we reach a coin showing heads, which cannot happen before we get past the coin with number k_i.

Since, obviously, $m_i \leq 8$ for all i, the inequality $m_{i+1} > m_i$ leads to an upper bound on the total number of turns: $m_1 + m_2 + m_3 + \cdots \leq 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$. This number is indeed achieved, for an initial configuration $TTTTHHHH$ (T for tails, H for heads).

Alternative solution: Let us consider a more general problem where we start with an arbitrary number n of coins. Let F_n be the maximal possible number of turns for n coins and W_n be an initial configuration for which this number is achieved. We denote a configuration of coins by a string of letters H ("heads") and T ("tails"). It is easy to see that $F_1 = 1$ and $W_1 = H$ ($H \to T$). Also, $F_2 = 3$ and $W_2 = TH$ ($TH \to HH \to HT \to TT$). Notice that W_1 and W_2 are unique. We are going to show that for any $n \geq 3$ we have $F_n = F_{n-2} + 2n - 1$ and $W_n = TW_{n-2}H$, or else $F_n = F_{n-1}$ and $W_n = W_{n-1}T$. Then it follows by induction that $F_n = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \geq 1$. In particular, $F_8 = 36$. Besides, it follows that the configuration W_n is unique, $\lfloor n/2 \rfloor$ tails followed by $\lceil n/2 \rceil$ heads. In particular, $W_8 = TTTTHHHH$.

Let W be an initial configuration of $n \geq 3$ coins. First we consider the case when $W = \boxed{}\,T$. In this case the rightmost coin is never turned over. All turns happen inside the box, moreover, they are done as if $n-1$ coins in the box was the complete configuration. Therefore we get the maximal possible number of turns (for this particular case) if the configuration in the box is W_{n-1}. It follows that $F_n \geq F_{n-1}$. Furthermore, if $F_n = F_{n-1}$ then $W_n = W_{n-1}T$.

Next we consider the case $W = T\boxed{}\,H$. In this case all turns happen inside the box until we get all tails in the box. Moreover, $n-2$ coins in the box are turned over as if that were the complete configuration. Therefore we get the maximal number of turns if the configuration in the box is W_{n-2}. Once we get all tails in the box, the full configuration is $n-1$ tails followed by one heads. After that all tails are turned over from left to right so that we get n heads. Then all coins are turned over one last time, this time from right to left, and we get all tails. It follows that $F_n \geq F_{n-2} + 2n - 1$. Furthermore, if $F_n = F_{n-2} + 2n - 1$ then $W_n = TW_{n-2}H$.

Similarly, we consider the case $W = HH \ldots HT\boxed{}\,H$. Let $k \geq 1$ be the number of heads to the left of the box. Then we have $n - k - 2$ coins in the box. The maximal number of turns is achieved when the initial configuration in the box is W_{n-k-2}. First we have F_{n-k-2} turns inside the box. Once we get all tails in the box, the full configuration is k heads followed by $n - k - 1$ tails followed by one heads. After that all tails are turned over from left to right so that we get n heads and, finally, all coins are turned over from right to left. The total number of turns is $F_{n-k-2} + 2n - k - 1$. Notice that $F_m \geq F_{m-1}$ for any $m \geq 2$ as follows from the above. Therefore $F_{n-k-2} + 2n - k - 1 \leq F_{n-2} + 2n - k - 1 < F_{n-2} + 2n - 1$. Hence in this case we can never get as many turns as with the initial configuration $TW_{n-2}H$.

The remaining possibilities for W are $HH \ldots HTH$ and $HH \ldots H$. They will lead to $n+1$ and n turns, respectively, which is less than $F_{n-2} + 2n - 1$. Thus $F_n = \max(F_{n-1}, F_{n-2} + 2n - 1)$. Moreover, $W_n = W_{n-1}T$ if $F_n = F_{n-1}$ and $W_n = TW_{n-2}H$ if $F_n = F_{n-2} + 2n - 1$.

17. Three solid balls of radius 1 are placed on a horizontal floor so that they touch one another. The balls are firmly attached to the floor and cannot move. The fourth ball of radius 1 is put into a hole formed by the first three balls. Find the clearance between the fourth ball and the floor.

Answer: $\sqrt{\dfrac{8}{3}} = 2\sqrt{\dfrac{2}{3}} = \dfrac{2\sqrt{6}}{3}$.

Let O_1, O_2 and O_3 be centers of the first three balls. Let D_i, $1 \le i \le 3$ be the point at which the ball with center O_i is attached to the floor. Let O_4 be the center of the fourth ball and D_4 be the point of this ball closest to the floor. The clearance d between the fourth ball and the floor is the distance from D_4 to the plane of the triangle $D_1D_2D_3$. Note that D_1O_1, D_2O_2, D_3O_3 and D_4O_4 are vertical segments of length 1. Therefore d is also the distance from the point O_4 to the plane of the triangle $O_1O_2O_3$.

It is easy to observe that the fourth ball touches the first three balls. Hence all four balls touch one another. It follows that the distance between any two of the centers is $1+1=2$. Let H be the projection of O_4 onto the plane of the triangle $O_1O_2O_3$. Then $d = |O_4H|$. The segment O_4H is orthogonal to O_iH for $1 \le i \le 3$. By the Pythagorean Theorem, $|O_iH|^2 = |O_iO_4|^2 - |O_4H|^2 = 4 - d^2$. In particular, H is the center of the equilateral triangle $O_1O_2O_3$ and O_iH, $1 \le i \le 3$ are radii of the circumscribed circle.

Let OA be the altitude of the triangle O_1HO_2. Then $|O_1H| = |O_1A|/\cos\angle HO_1O_2$. Since the triangle O_1HO_2 is isosceles, $|O_1H| = |O_2H|$, the altitude OA is also the median. Hence $|O_1A| = \frac{1}{2}|O_1O_2| = 1$. Further, the triangles O_1HO_2 and O_1HO_3 are congruent since $|O_1H| = |O_2H| = |O_3H|$ and $|O_1O_2| = |O_1O_3|$. It follows that $\angle HO_1O_2 = \angle HO_1O_3$. Therefore $\angle HO_1O_2 = \frac{1}{2}\angle O_3O_1O_2 = \pi/6$. Then $|O_1H| = 1/\cos(\pi/6) = 2/\sqrt{3}$. Since $|O_1H|^2 = 4 - d^2$, we obtain that $d^2 = 4 - |O_1H|^2 = 4 - 4/3 = 8/3$, then $d = \sqrt{8/3} = 2\sqrt{2/3} = 2\sqrt{6}/3$.

18. Find all integers n between 100 and 200 such that the number $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ (n factorial) is divisible by 2^{n-1}.

Answer: 128.

We are going to show that $n!$ is divisible by 2^{n-1} if and only if n is a power of 2. The only power of 2 in the range from 100 to 200 is $128 = 2^7$.

For any positive integers n and k let $N_k(n)$ denote the number of all integers in the range from 1 to n divisible by 2^k. Since $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, the largest power of 2 that divides $n!$ is 2^K, where $K = N_1(n) + N_2(n) + N_3(n) + \ldots$ It is easy to observe that $N_k(n) = \lfloor n/2^k \rfloor$. Let k_0 be the largest integer such that $2^{k_0} \le n$. Then $N_k(n) = 0$ for $k > k_0$. We obtain

$$K = \left\lfloor \frac{n}{2^1} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2^{k_0}} \right\rfloor \le \frac{n}{2^1} + \frac{n}{2^2} + \cdots + \frac{n}{2^{k_0}} = n\left(1 - \frac{1}{2^{k_0}}\right) = n - \frac{n}{2^{k_0}} \le n - 1.$$

Moreover, the equality is attained only if $n = 2^{k_0}$, that is, n is a power of 2.

19. A right triangle with sides of length 3, 4 and 5 is cut out of paper. One folds the triangle along a straight line so that the folded figure is also a triangle. What is the minimal possible area of the new triangle?

Answer: $\dfrac{10}{3}$.

Let ℓ be the line along which we fold the triangle. The line ℓ intersects the triangle in a segment, which is going to be a side of the folded polygon. If the segment connects two sides of the original triangle, then the sum of two angles adjacent to the segment will be greater than π, which implies that the folded polygon will not be a triangle. Hence the line ℓ goes through a vertex of the original triangle.

Let A be the vertex of the original triangle that lies on ℓ. Let B and C be the other two vertices denoted so that $|AB| > |AC|$. Let D be the point at which the line ℓ intersects the side BC. Folding consists of reflecting one of the triangles ADB and ADC (say, ADC) about ℓ. Note that points A, B and D are going to be vertices of the folded polygon. If $\angle BAD < \angle CAD$ then the reflected image of C will be another vertex and we will not get a triangle. On the other hand, if $\angle BAD \geq \angle CAD$ then the reflected image of the triangle ADC is contained inside the triangle ADB and so the folded figure is the triangle ADB. To minimize the area of this triangle, we need to minimize the angle BAD. The angle is minimal when $\angle BAD = \angle CAD$, that is, when we fold along the angle bisector. The area of the triangle ADB equals $\frac{1}{2}|AD| \cdot |AB| \sin \angle BAD$ and the area of the triangle ADC equals $\frac{1}{2}|AD| \cdot |AC| \sin \angle CAD$. In the case $\angle BAD = \angle CAD$, we obtain

$$\frac{\text{area}(\triangle ADB)}{\text{area}(\triangle ADC)} = \frac{|AB|}{|AC|}, \quad \text{then} \quad \frac{\text{area}(\triangle ADB)}{\text{area}(\triangle ABC)} = \frac{|AB|}{|AB| + |AC|}.$$

The triangle ABC has sides of length 3, 4 and 5. Since $3^2+4^2 = 5^2$, it is right, with legs 3 and 4. Hence $\text{area}(\triangle ABC) = \frac{1}{2} \cdot 3 \cdot 4 = 6$. Depending on which vertex is A, the ratio $|AB|/(|AB| + |AC|)$ can be $4/(4+3) = 4/7$, $5/(5+3) = 5/8$ or $5/(5+4) = 5/9$. The least of these numbers is $5/9$. Thus the minimal possible area of the new triangle is $6 \cdot 5/9 = 10/3$, achieved when folding along the bisector of the angle formed by the sides of length 4 and 5.

20. An integer-valued function $f(n)$ of an integer argument n satisfies a functional equation $f(f(x)) + 2f(y) = f(x+2y) - 3$ for all integers x and y. Find $f(5)$.

Answer: 4. [The function is $f(n) = n - 1$.]

Given an arbitrary integer z, let us write down the functional equation first for $x = 0$, $y = z+1$ and then for $x = 2$, $y = z$:

$$f(f(0)) + 2f(z+1) = f(2z+2) - 3,$$
$$f(f(2)) + 2f(z) = f(2z+2) - 3.$$

Since the right-hand sides are the same in both equalities, it follows that

$$f(f(0)) + 2f(z+1) = f(f(2)) + 2f(z).$$

Then $f(z+1) - f(z) = \alpha$, where $\alpha = \frac{1}{2}f(f(2)) - \frac{1}{2}f(f(0))$ is a constant. Hence for any integer n the sequence $f(n), f(n+1), f(n+2), \ldots$ is an arithmetic progression with common difference α. It follows that f is a linear function, $f(n) = \alpha n + \beta$ for all integers n, where β is another constant. Substituting this formula into the functional equation, we obtain

$$\alpha(\alpha x + \beta) + \beta + 2(\alpha y + \beta) = \alpha(x + 2y) + \beta - 3,$$

which is simplified to $(\alpha^2 - \alpha)x + \alpha\beta + 2\beta + 3 = 0$. Since x can be any integer, the functional equation is satisfied if and only if $\alpha^2 - \alpha = 0$ and $\alpha\beta + 2\beta + 3 = 0$. The first of the two equations has solutions $\alpha = 0$ and $\alpha = 1$. If $\alpha = 0$ then $2\beta + 3 = 0$ so that $\beta = -\frac{3}{2}$. If $\alpha = 1$ then $3\beta + 3 = 0$ so that $\beta = -1$. Thus $f(n) = -\frac{3}{2}$ or $f(n) = n - 1$. Since the function f is supposed to take integer values, the right formula is $f(n) = n - 1$. Then $f(5) = 4$.