CD EXAM

Texas A&M High School Math Contest

November 9th, 2019

Directions: Use exact numbers. For example, if your answer includes π, e, square root etc, do not replace it by an approximate value.

1. A positive integer n written in base b is 25_b. If $2n$ is 52_b, what is b?

   Sol.

$$\begin{aligned} \text{Solution: } n &= 2b + 5 \\ 2n &= 4b + 10 = 5b + 2 \\ b &= 8 \end{aligned}$$

   Ans. 8

2. Given that 23^{100} is a 137 digit number, find the number of digits of 23^{23}.

   Sol. Since 23^{100} is a 137 digit number, we have $136 \leq \log 23^{100} < 137$. This implies that

$$1.36 \leq \log 23 < 1.37.$$

   So 23^{23} is a 32 digit number because $31.28 = 23(1.36) \leq \log 23^{23} < 23(1.37) = 31.51$.
   Ans. 32

3. Let α and β be two solutions of $(x + 2020)^2 - (x + 2020) + 2019 = 0$. Find $(\alpha + 2019)(\beta + 2019)$.

   Sol. The equation can be written as

$$x^2 - (1 - 2 \cdot 2020)x + 2020^2 - 2020 + 2019 = 0.$$

   If α and β are the roots, we have

$$\alpha + \beta = 1 - 2 \cdot 2020 \quad \text{and} \quad \alpha\beta = 2020^2 - 1.$$

   Thus we have

$$\begin{aligned} (\alpha + 2019)(\beta + 2019) &= \alpha\beta + 2019(\alpha + \beta) + 2019^2 \\ &= 2020^2 - 1 + 2019(1 - 2 \cdot 2020) + 2019^2 \\ &= (2020 + 1)(2020 - 1) + 2019(1 - 2 \cdot 2020) + 2019^2 \\ &= 2019\Big(2021 + 1 - 2 \cdot 2020 + 2019\Big) \\ &= 2019 \cdot 1 = 2019. \end{aligned}$$

   Ans. 2019

4. Let P be the point $(3, 1)$. Let Q be the reflection of P across the x- axis, let R be the reflection of Q about the line $y = x$ and let S be the reflection of R through the origin. What is the area of the quadrilateral $PQSR$?

   Sol. We want to find the area of quadrilateral $PQSR$ for $P(3, 1)$, $Q(3, -1)$, $R(-1, 3)$ and $S(1, -3)$. We can cut the quadrilateral by a vertical line $x = 1$. The intersection of $\overline{PR}$ and $x = 1$ is $T(1, 2)$. With the base $ST = 5$ of both the trapezoid $TSQR$ and $\triangle TRS$, and $PQ = 2$, the area is

$$\frac{5 \cdot 2}{2} + \frac{2(2 + 5)}{2} = 5 + 7 = 12.$$

   Ans. 12

5. Assume that clock hands move continuously on the clock. Find the first (earliest) time and the last time when two hands overlap strictly between 12:00 AM and 12:00 PM. Write the answer as pairs (x, y), where x is hours and y is minutes.

   Sol. We read the angle clock-wise. Each minute increases the angle of minute hand by $\left(\frac{360}{60}\right)^\circ = 6^\circ$. Since 60 minutes contributes 30° to hour hand's angle, the hour hand moves $\left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2}\right)^\circ$ in each minute. Let θ_1 and θ_2 be the angle of hour hand and minute hand from 12 o'clock respectively. If the time is x hour and y minute,

$$\theta_1 = 30x + \frac{y}{2} \quad \text{and} \quad \theta_2 = 6y.$$

   We solve the equation $\theta_1 = \theta_2$ with the condition that $x = 0, 1, 2, \cdots, 11$. The first time is when $x = 1$ and $y = \frac{60}{11}$, the last time is when $x = 10$ and $y = \frac{600}{11}$.

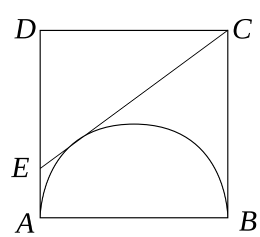
   Ans. $(x, y) = (1, \frac{60}{11})$ and $(x, y) = (10, \frac{600}{11})$

6. Let P be a point on the circle $x^2 + y^2 = 9$. Find the length of locus of the centroid of $\triangle PQR$ where $Q = (2, 5)$ and $R = (7, 4)$.

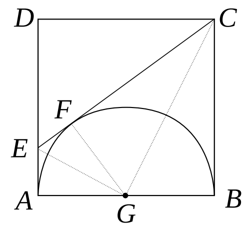
   Sol. If C is the centroid of $\triangle PQR$ and M is the middle of the segment QR, then $MC : MP = 1 : 3$. It follows that the locus is obtained from the circle by the dilation (homothety) with scale factor $1/3$ with center in M. Since the radius of the circle is 3, the locus is a circle of radius 1, hence its length is 2π.

   Ans. 2π

7. Square $ABCD$ has side length 2. A semicircle with diameter AB is constructed inside the square, and the tangent to semicircle from C intersects side AD at E. What is the length of CE?

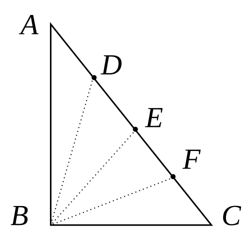


Sol. Let F be the point at which CE is tangent to the semicircle, and let G be the midpoint of AB as in the figure below. Because CF and CB are both tangents to the semicircle, $CF = CB = 2$. Similarly, $EA = EF$. Let $x = EA$. The Pythagorean Theorem applied to the triangle $\triangle CDE$ gives $(2-x)^2 + 2^2 = (2+x)^2$. It follows that $x = 1/2$ and $CE = 2 + x = 5/2$.



Ans. $\frac{5}{2}$

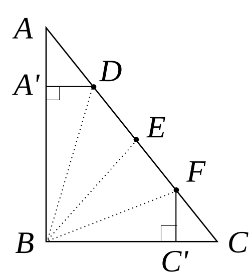
8. Consider a triangle $\triangle ABC$ with $\angle B = 90°$. Suppose the distances from B to the quadrisection points D, E and F of $\overline{AC}$ are $\cos x$, x and $\sin x$ respectively. Find x.



Sol. Observed that $\overline{EA}$, $\overline{EB}$ and $\overline{EC}$ are radii of the circumscribed circle of $\triangle ABC$. So $2x = AC$. To find AC, let $BA = a$ and $BC = c$. Draw lines $\overline{A'D}$ and $\overline{FC'}$ as in the figure below. Apply Pythagorean theorem to $\triangle BFC'$, $\triangle A'DB$ and $\triangle ABC$ to have

$$\left(\frac{a}{4}\right)^2 + \left(\frac{3c}{4}\right)^2 = \sin^2 x,$$
$$\left(\frac{3a}{4}\right)^2 + \left(\frac{c}{4}\right)^2 = \cos^2 x,$$
$$a^2 + c^2 = AC^2.$$

Adding the first two identities we have

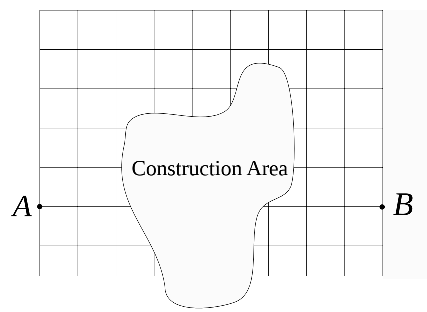


$$\frac{10}{16}(a^2+c^2)=1.$$

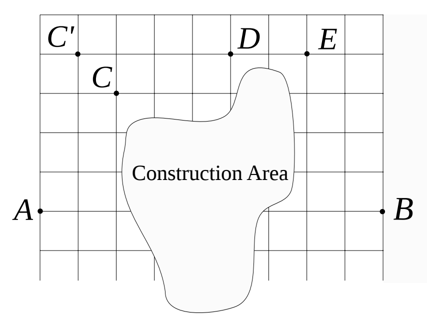
So we have $AC^2 = \frac{16}{10}$ or $AC = \frac{4}{\sqrt{10}} = \frac{4\sqrt{10}}{10}$. Thus $x = \frac{2\sqrt{10}}{10}$.

Ans. $\frac{\sqrt{10}}{5}$

9. The following map shows traffic system for two places A and B. Every square has side that equals 1 mile. Each car travels along horizontal and vertical grid lines. Find the number of shortest paths from A to B if one cannot cross the construction area.



Sol. Observe that each shortest path from A to B must pass D (see the figure below). There are only two types of paths from A to D: $A \to C \to D$ and $A \to C' \to D$. Let h and v denote moving horizontally and vertically by 1 unit respectively. The shortest path from A to C is determined by a word of length 5 with 2 h's and 3 v's. The number of such words is $\frac{5!}{2!\,3!} = 10$. Similarly there are $\frac{4!}{3!\,1!} = 4$ paths from C to D. Thus there are $10 \cdot 4 = 40$ shortest paths of type $A \to C \to D$. Since there is only one path from C' to D, there are $\frac{5!}{1!\,4!} = 5$ paths from A to D that pass C', and so $40 + 5 = 45$ paths from A to D. By the same manner, the number of paths $D \to E \to B$ is $1 \cdot \frac{6!}{2!\,4!} = 15$. Therefore the number of shortest paths from A to B is $45 \cdot 15 = 675$.



Ans. 675

10. Solve the equation $4 \cdot 9^{x-1} = 3\sqrt{2^{2x+1}}$.

Sol. The equation can be written as

$$2^2 \cdot 3^{2(x-1)} = 3 \cdot 2^{(2x+1)/2}.$$

Both sides of this equation are positive, thus we can apply $\log_2$ to both sides:

$$\log_2\left(2^2 \cdot 3^{2(x-1)}\right) = \log_2\left(3 \cdot 2^{(2x+1)/2}\right).$$

Using the properties of logarithms we get

$$\begin{aligned}
\log_2 2^2 + \log_2 3^{2(x-1)} &= \log_2 3 + \log_2 2^{(2x+1)/2} \\
2 + 2(x-1)\log_2 3 &= \log_2 3 + \frac{2x+1}{2} \\
2 + 2x\log_2 3 - 2\log_2 3 &= \log_2 3 + x + \frac{1}{2} \\
x(2\log_2 3 - 1) &= 3\log_2 3 - \frac{3}{2} \\
x(2\log_2 3 - 1) &= \frac{3}{2}\left(2\log_2 3 - 1\right) \\
x &= \frac{3}{2}.
\end{aligned}$$

Ans. $x = \dfrac{3}{2}$

11. The line $y = k$, $-1 < k < 0$, intersects two graphs $y = \sin x$ and $y = \cos x$ at four points ($0 \le x < 2\pi$). Let a, b, c and d be the x-coordinates of the intersections. Find

$$\sin\left(\frac{a+b+c+d}{4}\right) + \cos\left(\frac{a+b+c+d}{4}\right) + \tan\left(\frac{a+b+c+d}{4}\right).$$

Sol. Let a and b be the x-coordinates of the intersections of $y = k$ and $y = \sin x$, and let c and d be the x-coordinates of intersections of $y = k$ and $y = \cos x$. By the symmetry of $y = \sin x$ and $y = \cos x$, we see that

$$\frac{a+b}{2} = \frac{3\pi}{2} \quad \text{and} \quad \frac{c+d}{2} = \pi.$$

Thus the desired sum becomes

$$\sin\left(\frac{3\pi+2\pi}{4}\right) + \cos\left(\frac{3\pi+2\pi}{4}\right) + \tan\left(\frac{3\pi+2\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) + \tan\left(\frac{5\pi}{4}\right) = 1 - \sqrt{2}.$$

Ans. $1 - \sqrt{2}$

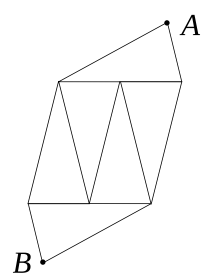
12. Find the number of subsets of $\{1, 2, 3, \cdots, 8\}$ that contain at least four consecutive numbers.

Sol. We can start with the subset $A = \{1, 2, 3, 4\}$. We may add or drop elements $5, 6, 7$ and 8 one by one to form a subset containing at least $1, 2, 3$ and 4. In this case we have 2^4 ways to form a subset containing A. Next consider a subset containing $\{2, 3, 4, 5\}$. If it contains 1, we already counted this subset. We can add elements $6, 7$ and 8 one by one. This yields 2^3 ways. Similarly, we can count the number of subsets that contains at least $\{3, 4, 5, 6\}$ by adding the remaining 3 elements $1, 7$ and 8. This also yields 2^3 ways. We have the same 2^3 ways to form a subset containing $\{4, 5, 6, 7\}$ or $\{5, 6, 7, 8\}$. Consequently the number of desired subsets is

$$2^4 + 2^3 + 2^3 + 2^3 + 2^3 = 48.$$

Ans. 48

13. In the figure below, there are six non-overlapping congruent isosceles triangles. The sides of each triangle are 2, 2 and 1. Find the distance from A to B.

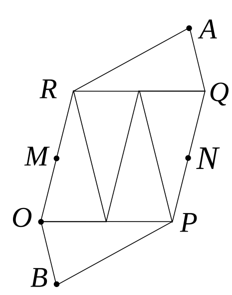


Sol. Let O be the origin, then $\overline{OP}$ is along the x-axis and so $P = (2, 0)$ in the figure below. Thus the x-coordinate of R is $1/2$, and the altitude y at R satisfies $y^2 + (1/2)^2 = 4$, which implies

$$R = \left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right) \text{ and } Q = \left(\frac{5}{2}, \frac{\sqrt{15}}{2}\right).$$

Let M and N be the midponts of $\overline{OR}$ and $\overline{PQ}$ respectively. Then

$$M = \left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right) \text{ and } N = \left(\frac{9}{4}, \frac{\sqrt{15}}{4}\right).$$



Since $\angle POB = \angle POR$ we see, by the symmetry, that the point B is the reflection of M about x-axis. Similarly, A is the reflection of N about $\overline{RQ}$. Thus

$$B = \left(\frac{1}{4}, -\frac{\sqrt{15}}{4}\right) \text{ and } A = \left(\frac{9}{4}, \frac{3\sqrt{15}}{4}\right).$$

Therefore $|AB| = \sqrt{19}$ since

$$|AB|^2 = \left(\frac{9}{4} - \frac{1}{4}\right)^2 + \left(\frac{3\sqrt{15}}{4} + \frac{\sqrt{15}}{4}\right)^2 = 19.$$

Ans. $\sqrt{19}$

14. Let $X = \{1, 2, \cdots, 10\}$. Find the number of one-to-one functions f with domain X and range X such that x and $f(x)$ are mutually prime for every x in X.

Sol. For every even number x, $f(x)$ must be odd. Being one-to-one, f maps distinct even numbers to distinct odd numbers. Consequently f maps the remaining 5 odd numbers to 5 even numbers. Moreover, f maps distinct odd numbers to distinct even numbers with the condition

$$f(3) \neq 6,\ f(5) \neq 10 \text{ and } f(9) \neq 6.$$

If $f(3) = 10$, then we have only three possibilities for $f(9)$, i.e., $f(9) = 2$, $f(9) = 4$ or $f(9) = 8$. There is no restriction for remaining odd numbers 1, 5 and 7 to be mapped to distinct remaining even numbers. The same counting yields that we have

$$2 \times 3 \times 3!$$

ways if either $f(3) = 10$ or $f(9) = 10$.

If $f(3) \neq 10$ and $f(9) \neq 10$, $f(3)$ and $f(9)$ must belong to $\{2, 4, 8\}$. Then $f(5)$ can be mapped to one of remaining 2 even numbers. There is no restriction for 1 and 7 be to mapped to the remaining two even numbers. We have

$$(3 \times 2) \times 2 \times 2!$$

ways for this case.

Observe that the restriction of f on the odd numbers determine one-to-one correspondence between $\{2, 4, 6, 8, 10\}$ and $\{1, 3, 5, 7, 9\}$. This means that there are precisely

$$2 \times 3 \times 3! + (3 \times 2) \times 2 \times 2! = 60$$

ways to define f restricted on the even numbers. Therefore the number of possible one-to-one functions $f : X \to X$ are

$$60^2 = 3600.$$

Ans. 3600

15. Find $a + b + c + d$ if four integers a, b, c and d satisfy the following conditions.

A: $10 \leq a, b, c, d \leq 20$

B: $ab - cd = 58$

C: $ad - bc = 110$

Sol. By taking the difference of two equations in B and C, we have

$$(ad - bc) - (ab - cd) = 110 - 58 \quad \Leftrightarrow \quad (a + c)(d - b) = 52.$$

Since $20 \le a+c \le 40$, $a+c$ must be 26 which is the only divisor of 52 between 20 and 40. Consequently $d - b = 2$. Plugging $c = 26 - a$ and $d = 2 + b$ back into B, we have

$$2ab - 26b + 2a - 52 = 58 \quad \text{or} \quad ab + a - 13b = 55.$$

Adding -13 to both sides we have

$$a(b + 1) - 13(b + 1) = 42 \quad \text{or} \quad (a - 13)(b + 1) = 42.$$

The condition A implies

$$11 \le b + 1 \le 21 \quad \text{and} \quad b + 2 = d \le 20.$$

Now $11 \le b + 1 \le 18$ implies that $b + 1 = 14$ which is the only divisor of 42 with that condition. Consequently $a - 13 = 3$. Therefore

$$a = 16,\ b = 13,\ c = 10 \text{ and } d = 15,$$

and so $a + b + c + d = 54$.

Ans. 54

16. Find the smallest number n such that the following statement is true. A collection of n points on the coordinate plane with integer coordinates contains a pair of points such that the trisection points of the line joining those two points have integer coordinates.

Sol. Let $P(a, b)$ and $Q(x, y)$ be two points with integer coordinates. The trisection point T of $\overline{PQ}$ that is closer to P is

$$T = \left(\frac{2a + x}{3}, \frac{2b + y}{3}\right)$$

Observe that T has integer coordinates if and only if $x \equiv a \pmod 3$ and $y \equiv b \pmod 3$, i.e., both $x-a$ and $y-b$ are multiples of 3. If $2a+x = 3k$ for some integer k, then $x-a = (2a+x)-3a = 3k-3a = 3(k - a)$. Conversely, if $x - a = 3k$ for some integer k, then $2a + x = 2a + (3k + a) = 3(k + 1)$, and so the x-coordinate of T is an integer. Similar argument proves the statement for the y-coordinate of T.

We want to find the smallest number n such that a collection of n integer points contains a pair of points where the differences in both coordinates are multiples of 3. Every integer, when divided by 3, leaves remainder 0, 1, or 2. This means that there are $3 \times 3 = 9$ types of remainder pairs in x- and y-coordinates. This implies that, if one takes a collection of 10 integer coordinate points, it contains at least a pair of points with the same type of remainder pairs, hence the differences in both coordinates are multiples of 3.

Ans. $n = 10$

17. Ninety nine people $p_1, p_2, \cdots, p_{99}$ shake hands with each other. It was observed that each person p_i shook hands with precisely i people for every i, $1 \leq i \leq 98$. Find the number of people that p_{99} shook hands.

Sol. The person p_{98} shook hands with 98 people. This means that p_{98} shook hands with all people except p_{98}. So p_1 shook hands with p_{98}, which is the only person with whom p_1 shook hands. This implies that p_1 didn't shake hands with p_{97}. We see that p_{97} shook hands all people but p_1 and p_{97}. Similarly, p_2 shook hands only with p_{98} and p_{97}. Consequently p_{97} shook hands with everyone but p_1 and p_{97}. By applying analogous argument we can check that all of $p_{98}, p_{97}, \cdots, p_{50}$ shook hands with p_{99} and that all of $p_1, p_2, \cdots, p_{49}$ didn't shake hands with p_{99}. Therefore p_{99} shook hands with 49 people.

Ans. 49

18. How many possible distinct integer solutions (a, b, c) does the equation have?

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = 1 \tag{1}$$

Sol. We may assume that $a > b > c$. Multiply by c and rewrite (1) as

$$\frac{c}{a} + \frac{c}{b} + \frac{1}{ab} = c - 1. \tag{2}$$

First observe that left hand side of (2) is positive and less than 3;

$$0 < \frac{c}{a} + \frac{c}{b} + \frac{1}{ab} < 1 + 1 + 1.$$

Since the right hand side of (2) is non-negative integer, the left hand side is either 1 or 2. Consequently $c = 2$ or $c = 3$.

Case I: $c = 2$. Plug $c = 2$ in (1) and multiply $2ab$ to have

$$2a + 2b + 1 = ab \quad \Rightarrow \quad a(b-2) - 2(b-2) = 5 \quad \Rightarrow \quad (a-2)(b-2) = 5.$$

Since 5 is prime and $a-2 > b-2$, we must have $a-2 = 5$ and $b-2 = 1$. This yields $(a, b, c) = (7, 3, 2)$.

Case II: $c = 3$. We have

$$\frac{3}{a} + \frac{3}{b} + \frac{1}{ab} = 2.$$

One can check that the left hand side is less than 2. Since $a \geq 5$ and $b \geq 4$, we have

$$\frac{3}{a} + \frac{3}{b} + \frac{1}{ab} \leq \frac{3}{5} + \frac{3}{4} + \frac{1}{20} = \frac{28}{20}.$$

This means that the given equation has no solution if $c = 3$.

Thus all possible triples (a, b, c) are

$$(7, 3, 2), (7, 2, 3), (3, 7, 2), (3, 2, 7), (2, 7, 3) \text{ and } (2, 3, 7).$$

Ans. 6

19. Let $x \neq 1$ be such that

$$\lfloor x \rfloor + \frac{2022}{\lfloor x \rfloor} = x^2 + \frac{2022}{x^2}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Find x^2.

Sol. Every real number x belongs to an interval $[t, t+1)$ for some integer t. Letting $\lfloor x \rfloor = t$, we can rewrite the given equation as

$$t + \frac{2022}{t} = x^2 + \frac{2022}{x^2} \quad \Rightarrow \quad (x^2)^2 - \left(t + \frac{2022}{t}\right)x^2 + 2022 = 0.$$

Thus $x^2 = t$ or $x^2 = \dfrac{2022}{t}$.

Case I. If $x^2 = \lfloor x \rfloor$, the only possible solution is $x = 1$ since $x \neq 0$.

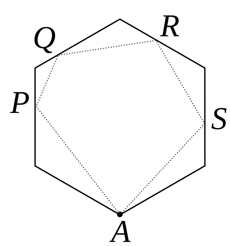
Case II. If $x^2 = \dfrac{2022}{\lfloor x \rfloor}$, then $x^2 \lfloor x \rfloor = 2022$. The inequality $12 \cdot 12^2 = 1728 < 2022 < 13 \cdot 13^2 = 2197$ suggests $\lfloor x \rfloor = 12$. Indeed,

$$(168.5)12 = 2022.$$

Thus we have $\lfloor x \rfloor = 12$ and $x^2 = 168.5$, which is the only solution. Therefore $x^2 = 168.5$ $(x \neq 1)$.

Ans. $x^2 = 168.5 = \dfrac{337}{2}$

20. Let A be a vertex of regular hexagon with side 1. Let P, Q, R and S be points on the four sides not containing A as in the figure. Find the minimum of $AP + PQ + QR + RS + SA$.

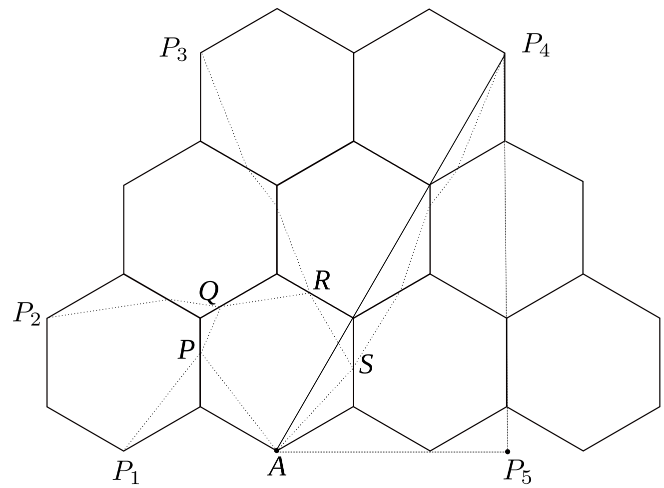


Sol. We can use reflections to find the shortest path. Let P_1 be the reflection of A about the vertical line containing P, as shown in the figure below. Since $AP = P_1P$, we have $AP + PQ = P_1P + PQ \geq P_1Q$. Applying similar inequalities we have

$$AP + PQ + QR \geq P_2Q + QR \geq P_2R,$$
$$AP + PQ + QR + RS \geq P_3R + RS \geq P_3S,$$
$$AP + PQ + QR + RS + SA \geq AS + SP_4 \geq AP_4$$

Indeed, the minimum is given by AP_4. To find AP_4, we apply the Pythagorean theorem to $\triangle AP_4P_5$. Since $AP_5 = 3\dfrac{\sqrt{3}}{2}$ and $P_4P_5 = 4 + \dfrac{1}{2}$, we have

$$AP_4^2 = \left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{9}{2}\right)^2 = \frac{27}{4} + \frac{81}{4} = \frac{108}{4} = 27.$$



Thus the minimum is $\sqrt{27} = 3\sqrt{3}$.
Ans. $3\sqrt{3}$

21. Find all integers $n \neq -1$ so that

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{2019}\right)^{2019}. \tag{3}$$

Sol. We first check that the equation has no positive integer solution. The equation (3) can be written as

$$\left(\frac{n+1}{n}\right)^{n+1} = \left(\frac{2020}{2019}\right)^{2019} \Leftrightarrow (n+1)^{n+1}2019^{2019} = n^{n+1}2020^{2019}. \tag{4}$$

Obviously $n = 2019$ is not a solution. If $n = 2019$, we must have

$$2020^{2020}2019^{2019} = 2019^{2020}2020^{2019} \Rightarrow 2020 = 2019.$$

Notice that the numbers n and $n+1$ have no common divisors greater than 1. Indeed, any common divisor of n and $n+1$ will also divide $(n+1)-n = 1$. Since 2019 and 2020 are mutually prime, 2019^{2019} in the left hand side of (4) must divide n^{n+1} in the right hand side. However, if $n \leq 2018$,

$$n^{n+1} \leq 2018^{2019} < 2019^{2019}.$$

This implies that the equation has no integer solution $0 < n \leq 2018$. Similarly, if $2020 \leq n$ then $2021^{2021} \leq (n+1)^{n+1}$, and so 2020^{2019} in the right hand side is not divisible by $(n+1)^{n+1}$. This contradicts that 2019^{2019} is an integer.

Next we consider integer solutions $n \leq -2$ $(n \neq -1, 0)$. Let $n = -k$ $(k \geq 2)$ and rewrite (3) to see

$$\left(1-\frac{1}{k}\right)^{1-k} = \left(1+\frac{1}{2019}\right)^{2019} \Leftrightarrow \left(1+\frac{1}{k-1}\right)^{k-1} = \left(1+\frac{1}{2019}\right)^{2019}. \tag{5}$$

Clearly $k = 2020$ satisfies (5) and so $n = -2020$ is a solution of (3) .

To show that equation (5) has only one solution $k = 2020$, we need to verify that the sequence $a_n = (1 + \frac{1}{n})^n$ is strictly increasing, i.e., $a_n < a_{n+1}$ for all $1 \leq n$. The inequality is obvious when $n = 1$. For $n \geq 2$, one can use binomial expansion directly to compare terms in a_n and a_{n+1};

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{n}{1!}\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \binom{n}{k}\left(\frac{1}{n}\right)^k + \cdots + \left(\frac{1}{n}\right)^n,$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} = 1 + \frac{n+1}{1!}\left(\frac{1}{n+1}\right) + \frac{(n+1)n}{2!}\left(\frac{1}{n+1}\right)^2 + \cdots + \binom{n+1}{k}\left(\frac{1}{n+1}\right)^k + \cdots + \left(\frac{1}{n+1}\right)^{n+1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The first two terms are identical in both expansions. For all $2 \leq k \leq n$, observe that the $(k+1)^{th}$ term in a_n can be written as

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right),$$

which is strictly less than the $(k+1)^{th}$ term of a_{n+1}

$$\frac{(n+1)n(n-1)\cdots(n-k+2)}{k!}\left(\frac{1}{n+1}\right)^k = \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right).$$

It follows that $a_n < a_{n+1}$. Therefore the equation has the only solution $n = -2020$.
Ans. $n = -2020$