BC Exam
Texas A&M High School Math Contest
November, 2022

Directions: *All answers should be simplified and if units are involved include them in your answer.*

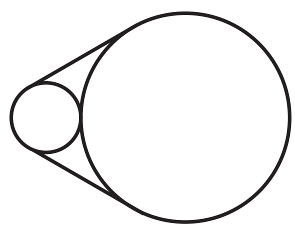
1. A speaker talked for sixty minutes to a full auditorium. Twenty percent of the audience heard the entire talk and ten percent slept through the entire talk. Half of the remainder heard one third of the talk and the other half heard two thirds of the talk. What was the average number of minutes of the talk heard by members of the audience?

2. A 25 foot ladder is placed against a vertical wall of a building. The foot of the ladder is 7 feet from the base of the building. How far (in feet) will the foot of the ladder slide, if the top of the ladder slips 4 feet?

3. All edges of a cube are divided into 4 equal parts by points. How many lines pass through pairs of these points (the vertices of the cube are not included)?

4. Find all non-zero solutions of the equation $20[x] - 22\{x\} = 0$, where $[x]$ is the largest integer not greater than x, and $\{x\} = x - [x]$.

5. A 6-inch and 18-inch diameter pole are touching each other as in the figure and are bound together with wire. Find the length of the shortest wire that will go around them.



6. Let H be the point of intersection of the heights of an acute triangle $\triangle ABC$, and suppose that $AB = CH$. Find $\angle ACB$.

7. An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

8. Two particles move along the edges of an equilateral triangle $\triangle ABC$ in the direction $A \to B \to C \to A$ starting simultaneously and moving with the same speed. One starts at A, and the other starts at the midpoint of $\overline{BC}$. The midpoint of the line segment joining the two particles traces out a path that encloses a region R. What is the ratio of the area of R to the area of $\triangle ABC$?

9. Suppose that x_0 and x_1, x_0 and x_2, $\ldots$, x_0 and x_n are the roots of the polynomials $x^2 + a_1x + b_1$, $x^2 + a_2x + b_2$, $\ldots$, $x^2 + a_nx + b_n$, respectively. Find, in terms of $x_0, x_1, \ldots, x_n$, the roots of the polynomial

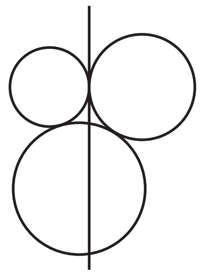
$$x^2 + \frac{a_1 + a_2 + \cdots + a_n}{n}x + \frac{b_1 + b_2 + \cdots + b_n}{n}.$$

10. In $\triangle ABC$, $AB = 5$, $BC = 7$, $AC = 9$ and D is on $\overline{AC}$ with $BD = 5$. Find the ratio $AD : DC$.

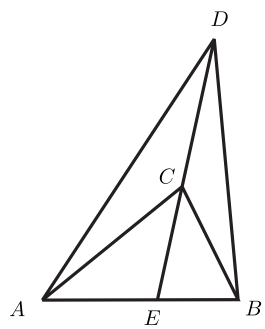
11. How many of the numbers 100, 101, ..., 999, have three digits in monotone (i.e., non-increasing or non-decreasing) order?

12. Find the sum of all natural numbers n such that n, $[n\sqrt{2}]$, $[[n\sqrt{2}]\sqrt{2}]$ is an arithmetic sequence. (Here $[x]$ is the largest integer not greater than x.)

13. Circles of radius 3, 4, and 5 are pairwise externally tangent, see the figure below. Consider the common tangent line of the circles of radii 3 and 4. What is the length of the segment of this line that is contained in the circle of radius 5?



14. Let $\overline{CE}$ be the bisector of the angle $\angle ACB$ of $\triangle ABC$. Let D be a point on the line $\overleftrightarrow{CE}$ such that C is between D and E and $\angle ADB = \angle ACE$. Find CD if $BC = a$ and $AC = b$.



15. Find all possible triples of non-zero digits x, y, z such that the decimal number $\overline{2022xyz}$ is divisible by 7, 8, and 9.

16. Find all decimal numbers $\overline{abc}$ such that $\overline{abc} = 2(\overline{ab} + \overline{bc} + \overline{ca})$ and $\overline{abc}$ is divisible by 9. (Here $\overline{abc}$ denotes the number $100a + 10b + c$, where a, b, c are digits from the set $\{0, 1, 2, \ldots, 9\}$, and $a \neq 0$.)

17. Find all pairs of natural numbers (n, m) such that $n! + 1 = (m! - 1)^2$.

18. Simplify $\frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \cdot \frac{4^3-1}{4^3+1} \cdots \frac{n^3-1}{n^3+1}$.

19. Let $x = \frac{b^2+a^2-c^2}{2ab}$, $y = \frac{a^2+c^2-b^2}{2ac}$, $z = \frac{b^2+c^2-a^2}{2bc}$. It is known that $x + y + z = 1$. Find all possible values of x.

20. How many quadruples (x, y, u, v) of positive integers are there satisfying the following system of equations

$$\begin{cases} x + y &= uv \\ u + v &= xy \end{cases}$$