BC Exam Solutions
Texas A&M High School Math Contest
November, 2022

1. Let x be the number of people in the audience. Then $x/5$ listened for 60 minutes, $x/10$ listened for 0 minutes. The remainder is $7x/10$, out of which $7x/20$ listened for 20 minutes, and $7x/20$ for 40 minutes. The average is

$$\left(60 \cdot \frac{x}{5} + 20 \cdot \frac{7x}{20} + 40 \cdot \frac{7x}{20}\right)/x = 12 + 7 + 14 = 33.$$

Answer: 33 minutes.

2. The distance from the bottom of the wall to the tip of the ladder is equal, by Pythagorian theorem to $\sqrt{25^2 - 7^2} = \sqrt{576} = 24$. If it slips 4 feet down, it will be 20 feet. Then the distance from the foot of the ladder to the wall will become $\sqrt{25^2 - 20^2} = 5\sqrt{5^2 - 4^2} = 15$, so it slides $15 - 7 = 8$ feet.

Answer: 8 feet.

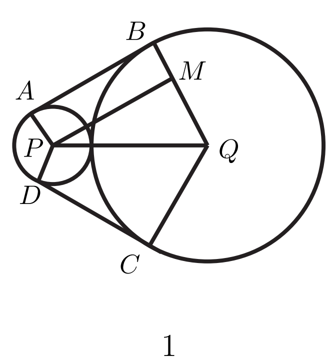
3. There are $3 \times 12 = 36$ points. They form $\frac{36 \cdot 35}{2}$ pairs. Each pair defines a line. These lines are different except for the lines passing through the points belonging to the same edge of the cube. Each edge has 3 points, which gives 3 pairs of points. So, we over-counted each line passing through an edge 2 times. It follows that there are $\frac{36 \cdot 35}{2} - 2 \cdot 12 = 18 \times 35 - 24 = 6(105 - 4) = 606$.

Answer: 606 lines.

4. We have $[x] = \frac{22}{20}\{x\}$ and $0 \le \{x\} < 1$, so we get $0 \le [x] < \frac{22}{20}$. Since $[x]$ is an integer, this implies that $[x] = 0$ or $[x] = 1$. In the first case, $\{x\} = 0$, so $x = 0$. In the second case, $\{x\} = \frac{20}{22} = \frac{10}{11}$, so $x = \frac{21}{11}$ is the only non-zero solution.

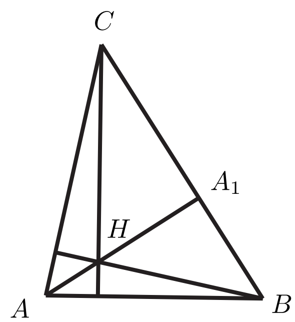
Answer: $x = \frac{21}{11} = 1\frac{10}{11}$.

5. The wire will consist of two segments $\overline{AB}$ and $\overline{CD}$ tangent to both circles (where A and D are on the pole of diameter 6, and B and C are on the pole of diameter 18). Let P and Q be the centers of the corresponding circles of diameter 6 and 18, respectively. Then $\angle PAB = \angle ABQ = 90°$. Let $ABMP$ be a rectangle. Then $MQ = 9 - 3 = 6$ and $PQ = 3 + 9 = 12$. We see that in the right triangle $\triangle PMQ$ the hypotenuse PQ is two times longer than the leg QM. Consequently, $\angle MQP = 60°$ and $AB = PM = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$. We also see that $\angle APD = 120°$, so the arc AP is equal to $6\pi/3 = 2\pi$. Similarly, $\angle BQC = 120°$, so the longer arc BC is equal to $\frac{2}{3}2\pi \cdot 9 = 12\pi$. It follows that the length of the wire is $12\sqrt{3} + 2\pi + 12\pi = 14\pi + 12\sqrt{3}$.



Answer: $14\pi + 12\sqrt{3}$ inches.

6. Denote by A_1 the foot of the height AA_1. Then $\triangle CHA_1$ is congruent to $\triangle ABA_1$, since $\angle HCB$ and $\angle HAB$ are both equal to $90° - \angle CBA$, and $AB = CH$. It follows that $CA_1 = AA_1$, so $\triangle ACA_1$ is an isosceles right triangle. It follows that $\angle ACA_1 = \angle ACB = 45°$.



Answer: $45°$.

7. If point (x, y) is the final point of a ten-step path, then $|x| + |y| \leq 10$ and the sum $x + y$ is even. On the other hand, it is easy to see that every point satisfying these conditions is a final point of a path (go to the point using the shortest path of $|x| + |y|$ steps, and then make the necessary number of pairs of consecutive left-right and up-down steps). Let us replace (x, y) by $(x + y, x - y)$. The condition $|x| + |y| \leq 10$ implies $|x + y| \leq 10$ and $|x - y| \leq 10$. Conversely, if $|x + y| \leq 10$ and $|x - y| \leq 10$, then $|x| + |y| \leq 10$, since $|x| + |y| \in \{x + y, -x + y, -x - y, x - y\}$, so $|x| + |y| = ||x| + |y|| \in \{|x + y|, |x - y|\}$. Note also that $x + y$ is even if and only if $x - y$ is even.

It follows that the number of points (x, y) such that $|x| + |y| \leq 10$ and $x + y$ is even is equal to the number of points (a, b) such that $|a| \leq 10, |b| \leq 10$ and a, b are even. There are 11 even numbers a such that $|a| \leq 10$.

Answer: 121 points.

8. Each of the particles has a constant velocity while it moves inside one side of $\triangle ABC$. If $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two particles moving with a constant velocity (v_1, u_1) and (v_2, u_2), respectively, then $x_1(t) = v_1 t + x_1(0)$, $y_1(t) = u_1 t + y_1(0)$, $x_2(t) = v_2 t + x_2(0)$, and $y_2(t) = v_2 t + y_2(0)$. Then the midpoint of the segment joining them is

$$\left(\frac{v_1 + v_2}{2}t + \frac{x_1(0) + x_2(0)}{2}, \frac{u_1 + u_2}{2}t + \frac{y_1(0) + y_2(0)}{2}\right).$$

Consequently, the midpoint moves with a constant speed, hence it traces a line segment.

The velocity of one of our particles changes when it passes through a vertex of $\triangle ABC$. Then the other particle is in the midpoint of the opposite side. It follows that the region R is the triangle formed by the midpoints of the medians of $\triangle ABC$. This triangle is obtained from $\triangle ABC$ by a homothety with center in the intersection point O of the medians. The coefficient of the homothety is $(\frac{2}{3} - \frac{1}{2}) : \frac{2}{3} = 1 : 4$. Consequently, the ratio of the area of the region R to the area of $\triangle ABC$ is $1/16$.

Answer: $1/16$.

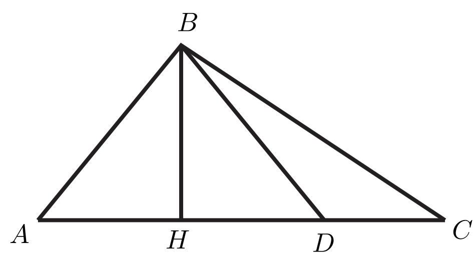
9. Let y_1 and y_2 be the roots of the last polynomial. Then $y_1 + y_2 = -\frac{a_1+a_2+\cdots+a_n}{n} = \frac{(x_0+x_1)+(x_0+x_2)+\cdots+(x_0+x_n)}{n} = x_0 + \frac{x_1+x_2+\cdots+x_n}{n}$.

We also have $y_1 y_2 = \frac{b_1+b_2+\cdots+b_n}{n} = \frac{x_0x_1+x_0x_2+\cdots+x_0x_n}{n} = x_0 \cdot \frac{x_1+x_2+\cdots+x_n}{n}$.

We see that roots y_1 and y_2 are x_0 and $\frac{x_1+x_2+\cdots+x_n}{n}$.

Answer: x_0 and $\frac{x_1+x_2+\cdots+x_n}{n}$.

10. Let BH be the height of $\triangle ABC$. Denote $AH = x$. Then square of the height is equal to $25 - x^2$ and to $49 - (9-x)^2$. So, $25 - x^2 = 49 - 81 + 18x - x^2$, which implies that $57 = 18x$, hence $x = \frac{19}{6}$. Since $\triangle ABD$ is isosceles, its height BH is also its median. Consequently, $AD = 2x = \frac{19}{3}$, and $DC = 9 - \frac{19}{3} = \frac{8}{3}$.



Answer: $AD : DC = 19 : 8$.

11. Let us count how many numbers there are with non-decreasing order of the digits. The middle digit can not equal to 0. If it is 1, then there is only one possibility for the first digit, and 9 for the last. If it is 2, then there are 2 possibilities for the first digit and 8 for the last. Continuing this way, we see that there are

$$1\cdot 9 + 2\cdot 8 + 3\cdot 7 + 4\cdot 6 + 5\cdot 5 + 6\cdot 4 + 7\cdot 3 + 8\cdot 2 + 9\cdot 1 = 2(9+16+21+24)+25 = 165.$$

We use similar arguments for the non-increasing order. In this case, if the middle digit is 0, then there are 9 possibilities for the first digit (any except for 0) and 1 for the last digit. If it is 1, then we still have 9 possibilities for the first digit, but 2 for the last one.

We see that we get the answer

$$9\cdot 1+9\cdot 2+8\cdot 3+7\cdot 4+6\cdot 5+5\cdot 6+4\cdot 7+3\cdot 8+2\cdot 9+1\cdot 10 = 9+2(18+24+28+30)+10 = 219.$$

Answer: 384 numbers.

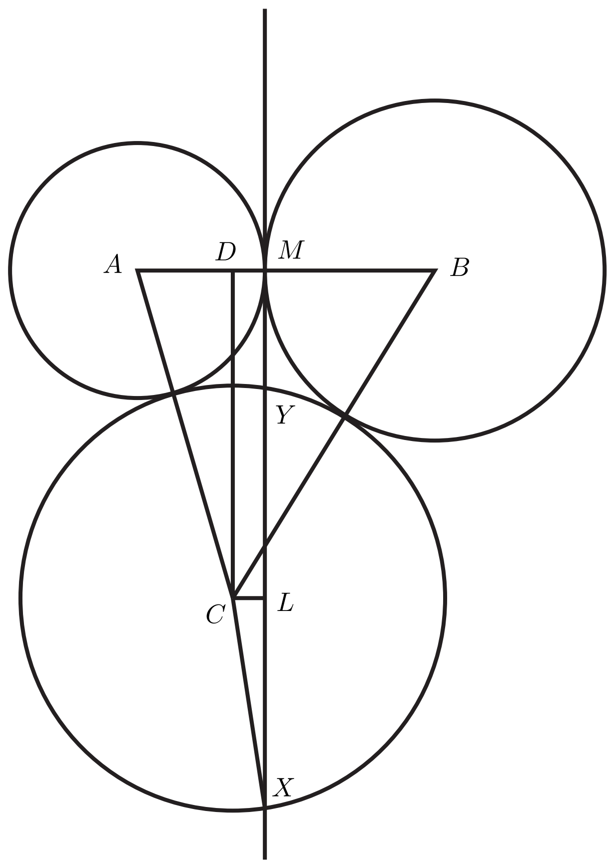
12. We are looking for n such that $n + [[n\sqrt{2}]\sqrt{2}] = 2[n\sqrt{2}]$. Since $[x] > x - 1$ for all x, we have $n + [n\sqrt{2}]\sqrt{2} - 1 < 2[n\sqrt{2}]$, hence $n + (\sqrt{2}-2)[n\sqrt{2}] < 1$. Since $\sqrt{2} - 2 < 0$ and $[n\sqrt{2}] < n\sqrt{2}$, we get $n + (\sqrt{2}-2)n\sqrt{2} < 1$, i.e., $(3-2\sqrt{2})n < 1$, which implies that $n < \frac{1}{3-2\sqrt{2}} = 3 + 2\sqrt{2} < 6$. It follows that $x \in \{1,2,3,4,5\}$.

For $n = 1$: $[n\sqrt{2}] = 1$, $[[n\sqrt{2}]\sqrt{2}] = 1$.
For $n = 2$: $[n\sqrt{2}] = 2$, $[[n\sqrt{2}]\sqrt{2}] = 2$.
For $n = 3$: $[n\sqrt{2}] = 4$ (since $16 < 18 < 25$), $[[n\sqrt{2}]\sqrt{2}] = 5$ (since $25 < 32 < 36$).
For $n = 4$: $[n\sqrt{2}] = 5$, $[[n\sqrt{2}]\sqrt{2}] = 7$ (since $49 < 50 < 64$).
For $n = 5$: $[n\sqrt{2}] = 7$, $[[n\sqrt{2}]\sqrt{2}] = 9$ (since $81 < 98 < 100$).

We see that all of them are arithmetic sequences, except for $n = 4$. It follows that the set of numbers satisfying the condition of the problem is $\{1,2,3,5\}$. Their sum is 11.

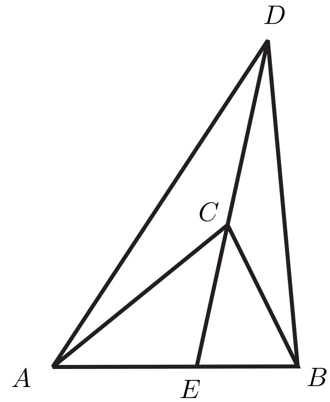
Answer: 11.

13. Let A, B, C be the centers of the circles of radii 3, 4, and 5, respectively. Let CD be the height of $\triangle ABC$. Denote $x = AD$. Then square CD is equal to $8^2 - x^2$ and to $9^2 - (7-x)^2$. It follows that $64 - x^2 = 81 - 49 + 14x - x^2$, so $x = \frac{16}{7}$. Let M be the common point of the circles of radii 3 and 4, and let L be the midpoint of the segment XY of the common tangent line to the circles of radii 3 and 4 contained in the circle of radius 5. Then $CLMD$ is a rectangle. Consequently, CL and DM are equal to $3 - x = \frac{5}{7}$. Then XL is equal to $\sqrt{5^2 - \frac{5^2}{7^2}} = \sqrt{\frac{25 \cdot 49 - 25}{49}} = \sqrt{\frac{25 \cdot 48}{49}} = \frac{20}{7}\sqrt{3}$. The length of XY is double of the length of XL, so $XL = \frac{40}{7}\sqrt{3}$.



Answer: $\frac{40\sqrt{3}}{7}$.

14. Since $\angle ECA = \angle CAD + \angle EDA = \angle BDA = \angle BDE + \angle EDA = \angle BCE = \angle CBD + \angle BDE$, we get that $\angle CAD = \angle BDE$ and $\angle CBD = \angle EDA$. Consequently, $\triangle CBD$ and $\triangle CDA$ are similar. It follows that $CD : a = b : CD$, so $CD = \sqrt{ab}$.



Answer: $\sqrt{ab}$.

15. A number is divisible by 7, 8, and 9 if and only if it is divisible by their product 504 (since they are pairwise coprime). The number $\overline{2022xyz} = 2,022,000 + \overline{xyz}$ is divisible by 504 if and only if $456 + \overline{xyz}$ is divisible by 504. We have $\frac{456+\overline{xyz}}{504} \leq \frac{1455}{504} < 3$. It follows that $\overline{xyz} = 504n - 456$ for $n = 1$ or $n = 2$. In the first case, $\overline{xyz} = 48$, i.e., $x = 0, y = 4, z = 8$, which is not allowed by the conditions of the problem. In the second case, $\overline{xyz} = 552$.

Answer: $(5, 5, 2)$.

16. We get $100a + 10b + c = 2(10a + b + 10b + c + 10c + a)$, so $100a + 10b + c = 22(a + b + c)$, so that $78a = 12b + 21c$. Dividing it by 3, we get $26a = 4b + 7c$. It follows that c is even, and we get $13a = 2b + \frac{7}{2}c$. The maximal value of the right-hand side is $2 \cdot 9 + \frac{7}{2} \cdot 8 = 46$. It follows that $a \in \{1, 2, 3\}$.

If $a = 1$, then we get $13 = 2b + \frac{7}{2}c$. Then $c < 4$, i.e., $c \in \{0, 2\}$. We see that $c = 2$ and $b = 3$, i.e., $\overline{abc} = 132$, but it is not divisible by 9;

If $a = 2$, then we get $26 = 2b + \frac{7}{2}c$. Since $2b$ is even, $\frac{7}{2}c$ must be also even, so $c \in \{0, 4\}$ (as $c = 8$ is too big). We can not have $c = 0$, since then $b = 13$. We see that $c = 4$, $b = 6$, i.e., $\overline{abc} = 264$ is the only solution in this case, but it is not divisible by 9;

If $a = 3$, then we get the equation $39 = 2b + \frac{7}{2}c$. Since $2b$ is even, $\frac{7}{2}c$ has to be odd, so $c \in \{2, 6\}$. The first case, $c = 2$, implies $b = 16$, which is not allowed. In the second case, we have $39 = 2b + 21$, so $b = 9$, and we get the solution $\overline{abc} = 396$. This is the only solutions that is divisible by 9.

Answer: 396.

17. The equation is equivalent to $n! = (m!)^2 - 2m!$, so $\frac{n!}{m!} = m! - 2$. Since $m \neq 0, 1, 2$, we get $n \geq m$, so

$$n(n-1)\cdots(m+1) = m! - 2.$$

If $n - m \geq 3$, then the left-hand side is divisible by 3, but $m! - 2$ is not. We get, therefore, $n \in \{m, m+1, m+2\}$ or , equivalently, $m \in \{n, n-1, n-2\}$. Now consider these three cases separately:

1. If $m = n$, then we get $1 = n! - 2$, i.e., $n! = 3$, which is impossible.

2. If $m = n - 1$, then $n = (n-1)! - 2$, i.e., $(n-1)! = n + 2$. We see that $n = 4$ is a solution, $n = 1, 2, 3$ are not, and $(n-1)! > n + 2$ for all $n > 4$.

3. If $m = n-2$, then $n(n-1) = (n-2)! - 2$ or, equivalently, $(n-2)! = n(n-1)+2$. Observe that $(n-2)! < n(n-1)+2$ for $n \le 6$, while $(n-2)! > n(n-1)+2$ for $n > 6$, so there are no solutions in this case.

Answer: $n = 4, m = 3$.

18. We have $(n+1)^2 - (n+1) + 1 = n^2 + n + 1$. Therefore, the product is equal to

$$\left(\frac{2-1}{2+1} \cdot \frac{3-1}{3+1} \cdot \frac{4-1}{4+1} \cdots \frac{n-1}{n+1}\right)\left(\frac{2^2+2+1}{2^2-2+1} \cdot \frac{3^2+3+1}{3^2-3+1} \cdot \frac{4^2+4+1}{4^2-4+1} \cdots \frac{n^2+n+1}{n^2-n+1}\right) = \frac{1 \cdot 2}{n(n+1)} \cdot \frac{n^2+n+1}{2^2-2+1} = \frac{2(n^2+n+1)}{3n(n+1)}.$$

Answer: $\frac{2(n^2+n+1)}{3n(n+1)}$

19. We have $(x-1)+(y-1)+(z+1) = 0$, so

$$\frac{b^2+a^2-c^2-2ab}{2ab} + \frac{a^2+c^2-b^2-2ac}{2ac} + \frac{b^2+c^2-a^2+2bc}{2bc} = 0.$$

This can be rewritten as

$$\frac{(a-b+c)(a-b-c)}{2ab} + \frac{(a-c+b)(a-c-b)}{2ac} + \frac{(b+c+a)(b+c-a)}{2bc} = 0$$

hence

$$0 = (b+c-a)\left(\frac{b-a-c}{2ab} + \frac{c-a-b}{2ac} + \frac{a+b+c}{2bc}\right) = (b+c-a)\frac{bc-ac-c^2+bc-ab-b^2+a^2+ab+ac}{2abc} = (b+c-a)\frac{2bc-c^2-b^2+a^2}{2abc} = \frac{(b+c-a)(a+b-c)(a-b+c)}{2abc}.$$

It follows that either $a = b+c$, or $c = a+b$, or $b = a+c$.

1. If $a = b+c$ or $b = a+c$, then $x - 1 = \frac{(a-b+c)(a-b-c)}{2ab} = 0$, so $x = 1$.

2. If $c = a+b$, then $x + 1 = \frac{(a+b+c)(a+b-c)}{2ab} = 0$, so $x = -1$.

If we take $a = 1, b = 1, c = 2$, then $x = -1$, $y = 1$, $z = 1$. If we take $a = 2, b = 1, c = 1$, then $x = 1$, $y = 1$, $z = -1$. This shows that the values $x = 1$ and $x = -1$ are possible.

Answer: $x = 1$ or $x = -1$.

20. Subtracting the equation $x + y = uv$ from $xy = u + v$, we get

$$xy - x - y = -uv + u + v,$$

i.e.

$$(x-1)(y-1) = -(u-1)(v-1) + 2.$$

Note that $(x-1)(y-1)$ is a nonnegative integer, and $-(u-1)(v-1)$ is a nonpositive integer. Their difference can be 2 in the following three cases:

1) $(x-1)(y-1) = 0$ and $(u-1)(v-1) = 2$. From the second equation, $(u, v) = (2, 3)$ or $(3, 2)$. From the first relation, $x = 1$ or $y = 1$. The initial equation $x + y = uv$ now implies that (x, y) is $(1, 5)$ or $(5, 1)$, and (u, v) is $(2, 3)$ or $(3, 2)$. We have 4 possible quadruplets in this case.

2) $(x-1)(y-1) = 1$ and $(u-1)(v-1) = 1$. Then $x = y = u = v = 2$ — the only possible quadruplet in this case.

3) $(x-1)(y-1) = 2$ and $(u-1)(v-1) = 0$. This case is similar to case 1), but (x, y) are switched with (u, v). Thus (u, v) is $(1, 5)$ or $(5, 1)$, and (x, y) is $(2, 3)$ or $(3, 2)$, so we have four possible quadruplets in this case.

Answer: 9 quadruples.